Lecture 04
Floating Point

CS213 – Intro to Computer Systems

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Slides adapted from:
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Administrivia

- Homework 1 due today! (11:59 pm Central)
  - Submit on Gradescope
  - About 55% of the class has submitted so far

- Pack Lab is out
  - Get started on this ASAP
  - If you still need a partner, I'll take last-chance requests today

- No Office Hours on Monday (01/20) for MLK Day holiday
  - Normal office hours on Friday and Tuesday though

Today's Goals

- Explore representing real (decimal) numbers with binary
- Understand IEEE754 encoding
- Discuss encoding impacts on floating-point arithmetic

What is hard about floating point?

- LOTS OF RULES
  - No, more than that

- Homework 2 will give you a chance to practice

- Plus on exams you’ll have a notes sheet to write down rules on

Outline

- **Fractional Binary Numbers**
- Representing Floating Point
- Smaller Floating Point
- Floating Point Arithmetic

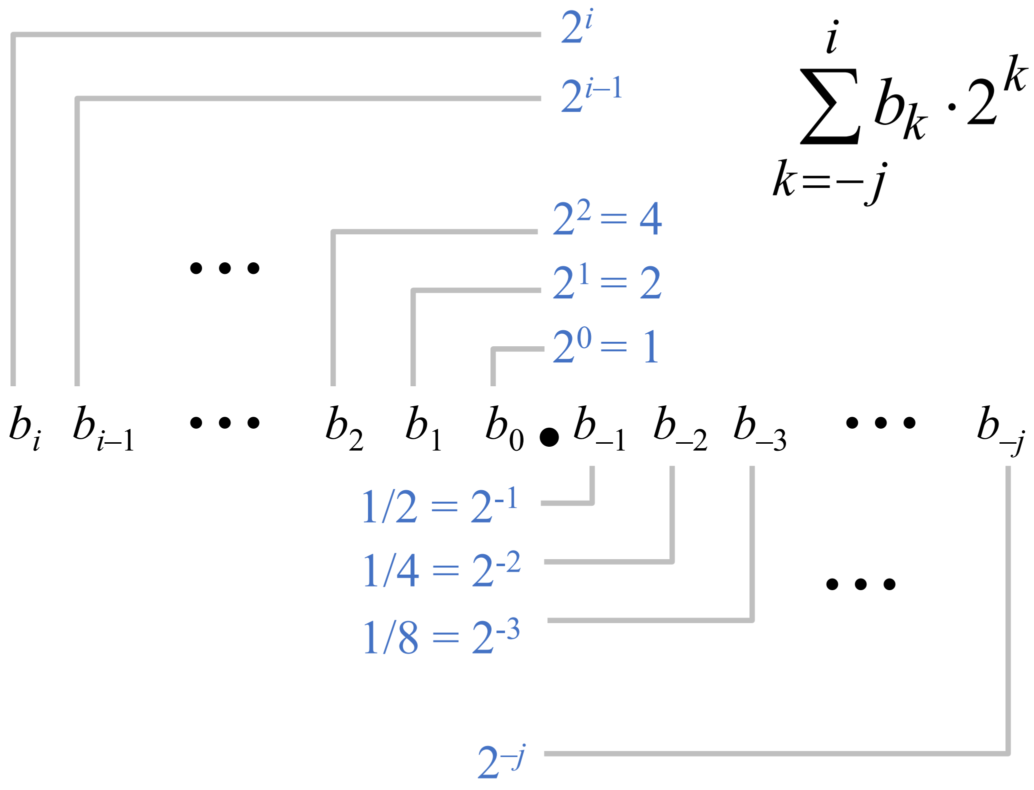
Floating point numbers

- In decimal:
  - 123450_{10}
  - 123.450_{10}
  - 1.23450_{10}

- We can use this same system in binary as well:
  - 1010110_2 (86_{10})
  - 1010.110_2 $(10.75_{10} = \frac{86}{2^3})$
  - 1.010110_2 $(1.34375_{10} = \frac{86}{2^6})$

Fractional Binary Numbers

- Representation
  - Bits to right of “binary point” represent fractional powers of 2
  - Represents rational number:



$$\sum_{k=-j}^{i} b_k \cdot 2^k$$

Example binary conversion

$\boxed{1010}.\boxed{110}$

Before the binary point:
$1*2^3 + 0*2^2 + 1*2^1 + 0*2^0 = 1*2^3 + 1*2^1 = 8+2 = 10$

After the binary point:
$1*2^{-1} + 1*2^{-2} + 0*2^{-3} = 1*2^{-1} + 1*2^{-2} = ½ + ¼ = ¾ = 0.75$

Fractional Binary Number Examples

- 5+3/4 = 0b101.$\boxed{11}$
- 2+7/8 = 0b10.$\boxed{111}$
- 63/64 = 0b0.$\boxed{111111}$

Note:

This the number 3!

This is the number 7!

This is the number 63!

Scientific Notation Binary

- Scientific notation works in binary just like in decimal

  - Decimal: $134 = 1.34 * 10^2$

  - Binary: $101 = 1.01 * 2^2$

- Positive and negative exponents change the direction the point moves
  - Positive powers get bigger: $1.00101 * 2^4$ -> 10010.1
  - Negative powers get smaller: $1.00101 * 2^{-3}$ -> 0.00100101

Binary point is part of the solution, but not an entire encoding

- Some problems remain:

1. Computers are finite, but real numbers are not
   - Need to choose how many bits to use
   - Many decimal numbers would take infinite binary bits to represent perfectly
     - $3.14_{10} = 11.0010001111010111_2$ (we could keep going)

2. We also need to represent where the “binary point” is located
   - We’ll use scientific notation and some of our bits to do so

3. Should do signed numbers while we’re at it

Outline

- Fractional Binary Numbers
- **Representing Floating Point**
- Smaller Floating Point
- Floating Point Arithmetic

Floating Point Standard – IEEE754

- Floating point representations
  - Encodes rational numbers of the form $V = m \times 2^e$
  - Base 2 scientific notation!

- IEEE Standard 754 (IEEE floating point)
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Headed by William Kahan, CS prof. at UC Berkeley (later won Turing Award for it)
  - Supported by all major CPUs

- Driven by numerical concerns and numerical analysts
  - Nice standards for rounding, overflow, underflow
  - Had to be implementable in fast hardware as well and support many languages

Flow for floating point translations

- Translating between decimal and IEEE754 floating point has a clear set of steps that you always take

- Steps from Decimal to IEEE754:
    1. Decimal
    2. Floating point binary
    3. Scientific notation binary
    4. IEEE754 encoding

- Steps for IEEE754 to Decimal:
    1. IEEE754 encoding
    2. Scientific notation binary
    3. Floating point binary
    4. Decimal

Floating Point Representation

Numerical form:

$$V = (-1)^{S} * M * 2^{E}$$

Exponent

Sign bit

Significand (Mantissa)

- Sign bit ***S*** determines whether number is negative or positive
- Significand ***M*** normally a fractional value in range [1.0,2.0) or [0.0,1.0)
  - Called ***mantissa*** or ***significand***
- Exponent ***E*** weights value by power of two

IEE754 Floating Point Encoding

Numerical form:

Exponent

$$V = (-1)^s * M * 2^E$$

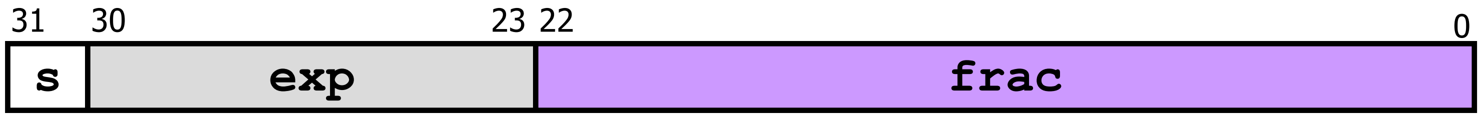
Sign bit Significand (Mantissa)

- IEEE754 Encoding
  - MSb is sign bit (can still look at most-significant bit alone to determine sign!)
  - **`exp`** field encodes E, *k*-bits (note: "*encodes E*" *!=* "*is E*")
  - **`frac`** field encodes M, *n*-bits

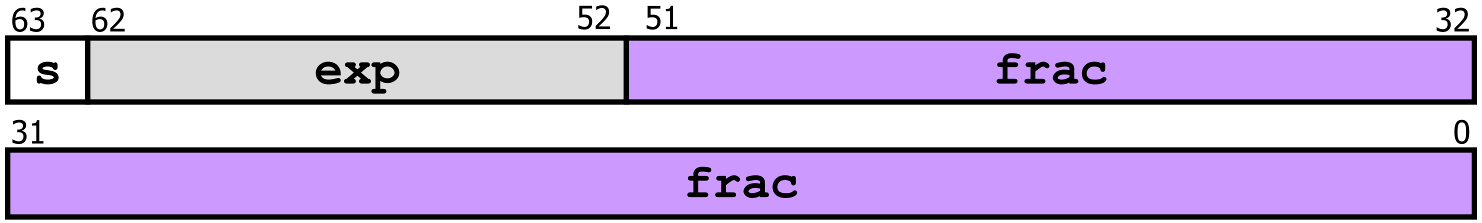
s	exp	frac

IEEE754 Floating Point Precision

- Sizes
  - Single precision: k = 8 exp bits, n= 23 frac bits (32b total). **`float`** in C



  - Double precision: k = 11 exp bits, n = 52 frac bits (64b total). **`double`** in C



Categories for IEEE754 Encoded Values

- Value encoded – three cases, depending on value of **`exp`**
  1. Normalized, the most common

s	**≠ 0 && not all 1s**	**`frac`**

  2. Denormalized (very small values)

s	**00000000**	**`frac`**

  3. Special values – infinity and NaN

Infinity	s	**11111111**	00000000000000000000000
NaN	s	**11111111**	**≠0**

Categories for IEE754 Encoded Values

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NaN	s	**11111111**	≠0

Normalized, Signifcand

$$V = (-1)^s * M * 2^E$$

- <u>Condition: not a special exponent (all zeros or ones)</u>

- Significand is encoded with implied leading 1
  - M = $1.xxx\ldots x_2$ ($1+f$ where $f = 0.xxx_2$)
    - xxx…x: bits of **`frac`** used directly

- Idea: every normalized number is 1.xxxx
  - So we're not going to include the leading 1 in the frac
  - We'll just know it's there when we convert to decimal
  - Saves one extra bit in the encoding!

`s`	**`exp`**	**`frac`**

Normalized, Exponent

$$V = (-1)^s * M * 2^E$$

- <u>Condition: not a special exponent (all zeros or ones)</u>

- Exponent coded as a biased value
  - E = Exp – Bias
    - Exp : unsigned value denoted by **`exp`**
    - Bias : Bias value = 2^{k-1} - 1, *k* is number of exponent bits
      - Single precision (8-bit exp): 127 (Exp: 1…254, E: -126…127)
      - Double precision (11-bit exp): 1023 (Exp: 1…2046, E: -1022…1023)

- Exponent really just pushes the binary point around
  - 1.11 * 2^2 = 11.1 * 2^1 = 111.0 * 2^0 = 111
  - 111 * 2^-2 = 11.1 * 2^-1 = 1.11 * 2^0 = 1.11

s	**exp**	**frac**

Decoding example for normalized floating point (32-bit)

- 0xC1900000 = 0b$\boxed{1}\boxed{10000011}\boxed{00100000000000000000000}$
  - Group bits **`s`**: 1 **`exp`**: 10000011 **`frac`**: 00100000000000000000000
  - **`exp`** is not all zeros or all ones => not a special case

- M = **1**.00100000000000000000000 = 1.001

- E = **`exp`** – **`bias`** = 131 – 127 = 4
  - bias = 2^{k-1} -1, k=8 -> 2^7-1 = 127

- Result = $(-1)^1 * 1.001_2 * 2^4 = -10.01_2 * 2^3 = -10010._2$ = -18

$$V = (-1)^s * M * 2^E$$

s	exp	frac

Normalized Encoding Example

- **Value**
  - **`float F = 15213.0; // single precision: 8 exp bits, 23 frac bits`**
  - $15213_{10} = 11101101101101_2 = 1.1101101101101_2 \times 2^{13}$
- **Significand**
  - $M = 1.\underline{1101101101101}_2$
  - $\text{frac} = \underline{1101101101101}0000000000$ pad with 0s ***on the right***. (example: 1.5 = 1.500)
- **Exponent**
  - $E = 13$
  - Bias = 127
  - $\text{exp} = E + \text{Bias} = 140 = 10001100_2$

More examples and practice in the bonus slides after the end

Floating Point Representation:

Hex:	4	6	6	D	B	4	0	0
Binary:	0100	0110	0110	1101	1011	0100	0000	0000
exp:	100	0110	0					
frac:			110	1101	1011	0100	0000	0000

Normalized Numbers: Why These Choices?

- Significand coded with **implied leading 1**
  - Any non-zero integer will start with a 1 bit somewhere
  - Leading 1 carries no information, so don't need to store it!
  - Can express mantissas between:
    - 1.0 when frac is all 0s
    - 2.0 (nearly) when frac is all 1s
      - Want smaller? Use a smaller exponent!

- Exponent coded as biased value
  - $E = Exp - Bias$
  - Alternative to using two's complement to represent signed integers
  - Reasons are a bit tricky
    - Floating point binary values increase in the same order as unsigned, which means they can be compared just like integer encodings
    - Bias also provides a more useful range (when considering denormalized)

Question + Break

- 0x3F800000 = 0b00111111100000000000000000000000
  - Group bits **`s`**: 0 **`exp`**: 01111111 **`frac`**: 00000000000000000000000
  - **`exp`** is not 0…0 or 1…1 => not a special case

- M =

- E = **`exp`** – **`bias`** =
  - bias = 2^{k-1} -1, k=8 -> 2^{7}-1 = 127

$$V = (-1)^{s} * M * 2^{E}$$

s	exp	frac

Question + Break

- 0x3F800000 = 0b00111111100000000000000000000000
  - Group bits **`s`**: 0 **`exp`**: 01111111 **`frac`**: 00000000000000000000000
  - **`exp`** is not 0…0 or 1…1 => not a special case

- M = 1.00000000000000000000000 = 1.0

- E = **`exp`** − **`bias`** = 127 − 127 = 0
  - bias = 2^{k-1} -1, k=8 -> 2^7-1 = 127

- Result = $(-1)^0 * 1.0_2 * 2^0 = 1$

$$V = (-1)^s * M * 2^E$$

s	exp	frac

Categories for IEE754 Encoded Values

- Value encoded – three cases, depending on value of **`exp`**
  1. Normalized, the most common

`s`	**≠ 0 && not all 1s**	**`frac`**

  2. **Denormalized** (very small values)

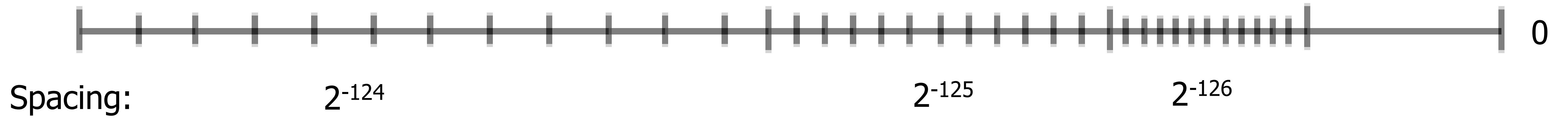
`s`	**00000000**	**`frac`**

  3. Special values – infinity and NaN

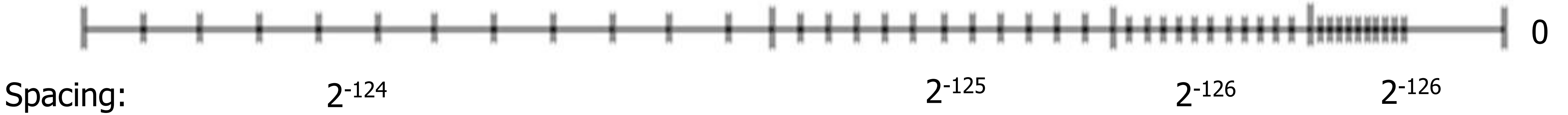
Infinity	**`s`**	**11111111**	00000000000000000000000
NaN	**`s`**	**11111111**	≠0

Normalized floating point leaves a gap around zero

- Gap is the size of $1.0000 * 2^{\text{Min Exponent}}$ (due to leading 1 bit)
  - And how do we encode "zero" anyways?

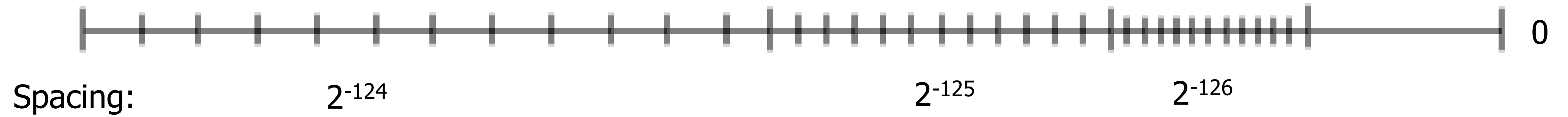


- Problem: if we just kept doing what we're doing we never get to 0
  - We keep getting half-way there

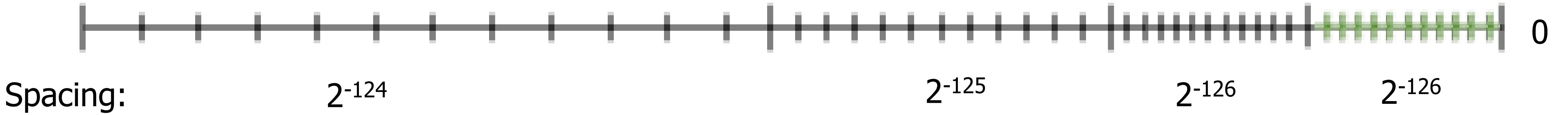


Solution is to do something different for the smallest numbers

- Gap is the size of 1.0000 * $2^{\text{Min Exponent}}$ (due to leading 1 bit)
    - And how do we encode "zero" anyways?



- Solution: fill in numbers between 0 and 1 * $2^{\text{Min Exponent}}$
    - Using same spacing as the previous range, in the form **<u>0</u>**.XXXXX



Denormalized Values

$$V = (-1)^s * M * 2^E$$

- Purpose: gracefully represent numbers approaching ±0
- Condition: exp = $000\ldots0_2$

- Value
  - Exponent value E = **<u>1 - Bias</u>**
    - Note: not simply E = 0 - Bias as it would be if we followed the previous rules
    - This means we're re-using the spacing from smallest normalized numbers
  - Significand value M = **<u>0</u>**.$\text{xxx}\ldots\text{x}_2$ (0.*frac*)
    - xxx…x: bits of frac. Leading 0 instead of leading 1

- Cases
  - exp = 000…0, frac = 000…0 => Represents value 0
    - Note that we have distinct values +0 and –0
  - exp = 000…0, frac $\neq$ 000…0 => Numbers very close to 0.0

Categories for IEE754 Encoded Values

- Value encoded – three cases, depending on value of **`exp`**
  1. Normalized, the most common

s	≠ 0 && not all 1s	frac

  2. Denormalized (very small values)

s	00000000	frac

  3. **Special values – infinity and NaN**

Infinity	s	11111111	00000000000000000000000
NaN	s	11111111	≠0

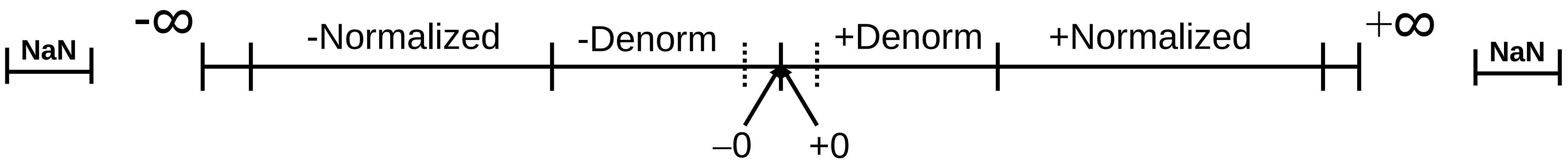
Special Values

- Purpose: represent quantities that $(-1)^s * M * 2^E$ cannot
- Condition: exp = $111\ldots1_2$
- Cases
  - exp = $111\ldots1_2$, frac = $000\ldots0_2$
    - Represents value ∞ (infinity)
    - Both positive and negative infinity (sign bit to tell apart)
    - Operation that overflows: nicer mathematical behavior than modulo!
    - E.g., 1.0/0.0 = -1.0/-0.0 = +∞, -1.0/0.0 = -∞
  - exp = $111\ldots1_2$, frac $\neq$ $000\ldots0_2$
    - Not-a-Number (NaN)
    - Represents case when no numeric value can be determined
      - Fraction could be used to distinguish sources (rarely used in practice)
    - E.g., $\sqrt{-1}$, ∞ - ∞, ∞ * 0

Floating Point in C

- C guarantees two levels
  - **`float`** single precision
  - **`double`** double precision
- Conversions
  - **`int`** → **`float`**
    - maybe rounded
    - less bits for actual value (32 → 23)
  - **`int`** or **`float`** → **`double`**
    - exact value preserved
    - double has greater range and higher precision (52 bits for **`frac`**)
  - **`double`** → **`float`**
    - may overflow, underflow (too small to represent), or be rounded (IEEE 754)
    - C99 standard says **undefined** if value out of range
  - **`double`** or **`float`** → **`int`**
    - rounded toward zero (-1.999 → -1)
    - C99 standard says **undefined** if value out of range

Break + Summary of FP Real Number Encodings



$$V = (-1)^s * M * 2^E$$

	Normalized	Denormalized
s	0/1 means +/-	0/1 means +/-
exp	exp $\neq 000\ldots0_2$ and exp $\neq 111\ldots1_2$	exp = $000\ldots0_2$
frac	$x_1x_2x_3\ldots x_j$	$x_1x_2x_3\ldots x_j$
Bias=	$2^{(k-1)} - 1$, for k exponent bits	$2^{(k-1)} - 1$, for k exponent bits
E=	exp – Bias	1 – Bias
M=	1. $x_1x_2x_3\ldots x_j$ a.k.a. 1.frac	0. $x_1x_2x_3\ldots x_j$ a.k.a. 0.frac
V=	$(-1)^s \times (1.\text{frac}) \times 2^{(\text{exp} - \text{Bias})}$	$(-1)^s \times (0.\text{frac}) \times 2^{(1 - \text{Bias})}$

Outline

- Fractional Binary Numbers
- Representing Floating Point
- **Smaller Floating Point**
- Floating Point Arithmetic

Floating point examples

- We'll often do floating point in custom bit widths
  - Rather than 32-bit (float) or 64-bit (double)

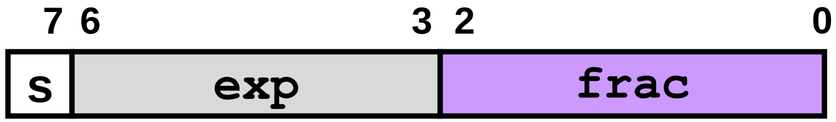
- Reasons
  1. 64 is just too many bits to write out and think about

  2. Make sure you understand the concepts of floating point
     - Smaller versions still demonstrate concepts! (e.g., 8-bit)

Example: Tiny Floating Point

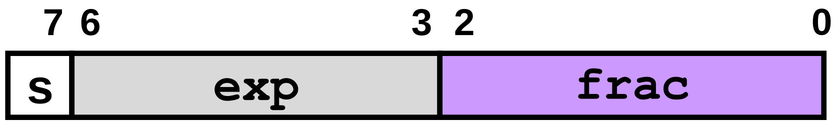
- 8-bit Floating Point Representation
  - Sign bit is in the most significant bit.
  - Next four (k) bits are exp, with a bias of 7 ($2^{k-1}-1$)
  - Last three (n) bits are frac

- Same general form as IEEE 754 format
  - normalized, denormalized numbers
  - representation of 0, NaN, infinity



Sidebar: increasingly useful for Machine Learning use!
- Models often don't need 32-bits of precision

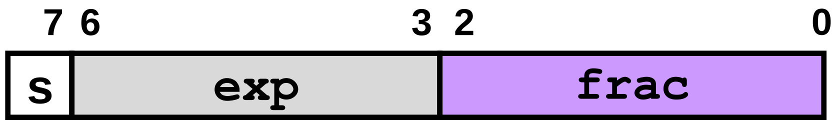
Denormalized encoding example



- Convert 5/512 to 8-bit tiny float
  - 5/512 = 0b101 * 2^{-9} = 10.1 * 2^{-8} = 1.01 * 2^{-7}

- E = exp - bias -> -7 = exp – ($2^{(4-1)}$-1) -> -7 = exp – 7
  - exp = 0 ???
  - But exp can’t be less than 1 (or we’re denormalized)
  - So, the answer must be a denormalized number. Reset the problem!

Denormalized encoding example



- Convert 5/512 to 8-bit tiny float
  - $5/512 = 0b101 * 2^{-9} = 10.1 * 2^{-8} = 1.01 * 2^{-7}$

- $E = 1 – bias = 1 – 7 = -6$

- $0.xxx * 2^{-6}$ -> $1.01 * 2^{-7} = 0.101 * 2^{-6}$

- S: 0 (positive) exp: 0000(denorm) frac: 101
- 0b0 0000 101 -> 0x05

Exponents for 8-bit tiny floats

Bias = $2^{4-1} - 1 = 7$
(4-bit exp)

Denormalized
E = 1 - Bias

Normalized
E = exp – Bias

Special

exp	exp	E	2^E	
0	0000	-6	1/64	(denorms)
1	0001	-6	1/64	
2	0010	-5	1/32	
3	0011	-4	1/16	
4	0100	-3	1/8	
5	0101	-2	1/4	
6	0110	-1	1/2	
7	0111	0	1	
8	1000	+1	2	
9	1001	+2	4	
10	1010	+3	8	
11	1011	+4	16	
12	1100	+5	32	
13	1101	+6	64	
14	1110	+7	128	
15	1111	n/a		(inf, NaN)

Dynamic Range of 8-bit tiny float

```
0 0000 000
0 0000 001
0 0000 010
...
0 0000 110
0 0000 111
0 0001 000
0 0001 001
...
0 0110 110
0 0110 111
0 0111 000
0 0111 001
0 0111 010
...
0 1110 110
0 1110 111
0 1111 000
0 1111 001
...
0 1111 111
```

Dynamic Range of 8-bit tiny float

```
s exp  frac

0 0000 000
0 0000 001
0 0000 010
...
0 0000 110
0 0000 111
0 0001 000
0 0001 001
...
0 0110 110
0 0110 111
0 0111 000
0 0111 001
0 0111 010
...
0 1110 110
0 1110 111
0 1111 000
0 1111 001
...
0 1111 111
```

Dynamic Range of 8-bit tiny float

Bias = 7

V= $(-1)^s \times (0.\text{frac}) \times 2^{(1 - \text{Bias})}$

Denormalized numbers

Normalized numbers

V= $(-1)^s \times (1.\text{frac}) \times 2^{(\text{exp} - \text{Bias})}$

Special values

```
s exp  frac

0 0000 000
0 0000 001
0 0000 010
...
0 0000 110
0 0000 111
-----------
0 0001 000
0 0001 001
...
0 0110 110
0 0110 111
0 0111 000
0 0111 001
0 0111 010
...
0 1110 110
0 1110 111
-----------
0 1111 000
0 1111 001
...
0 1111 111
```

Dynamic Range of 8-bit tiny float

Bias = 7

V= $(-1)^s \times (0.\text{frac}) \times 2^{(1 - \text{Bias})}$

Denormalized numbers

Normalized numbers

V= $(-1)^s \times (1.\text{frac}) \times 2^{(\text{exp} - \text{Bias})}$

Special values

s	exp	frac	*E*	Value
0	0000	000	-6	0
0	0000	001	-6	1/8*1/64 (2^{-6}) = 1/512
0	0000	010	-6	2/8*1/64 = 2/512
...				
0	0000	110	-6	6/8*1/64 = 6/512
0	0000	111	-6	7/8*1/64 = 7/512
0	0001	000	-6	8/8*1/64 = 8/512
0	0001	001	-6	9/8*1/64 = 9/512
...				
0	0110	110	-1	14/8*1/2 = 14/16
0	0110	111	-1	15/8*1/2 = 15/16
0	0111	000	0	8/8*1 = 1
0	0111	001	0	9/8*1 = 9/8
0	0111	010	0	10/8*1 = 10/8
...				
0	1110	110	7	14/8*128 = 224
0	1110	111	7	15/8*128 = 240
0	1111	000	n/a	inf
0	1111	001	n/a	NaN
...				
0	1111	111	n/a	NaN

Dynamic Range of 8-bit tiny float

Bias = 7

$V = (-1)^s \times (0.\text{frac}) \times 2^{(1 - \text{Bias})}$

Denormalized numbers

Normalized numbers

$V = (-1)^s \times (1.\text{frac}) \times 2^{(\text{exp} - \text{Bias})}$

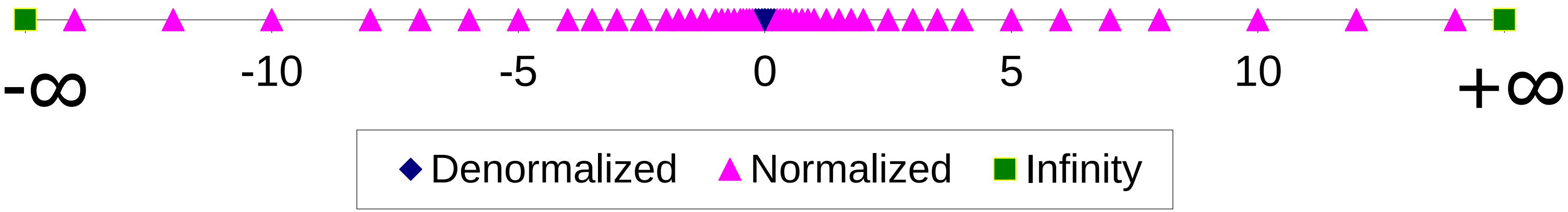
Special values

s	exp	frac	*E*	Value	Notes of Interest
0	0000	000	-6	0	
0	0000	001	-6	1/8*1/64 (2^{-6}) = 1/512	closest to zero
0	0000	010	-6	2/8*1/64 = 2/512	
...					
0	0000	110	-6	6/8*1/64 = 6/512	
0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
0	0001	000	-6	8/8*1/64 = 8/512	smallest norm > 0
0	0001	001	-6	9/8*1/64 = 9/512	
...					
0	0110	110	-1	14/8*1/2 = 14/16	
0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
0	0111	000	0	8/8*1 = 1	
0	0111	001	0	9/8*1 = 9/8	closest to 1 above
0	0111	010	0	10/8*1 = 10/8	
...					
0	1110	110	7	14/8*128 = 224	
0	1110	111	7	15/8*128 = 240	largest norm
0	1111	000	n/a	inf	
0	1111	001	n/a	NaN	
...					
0	1111	111	n/a	NaN	

Distribution of Values

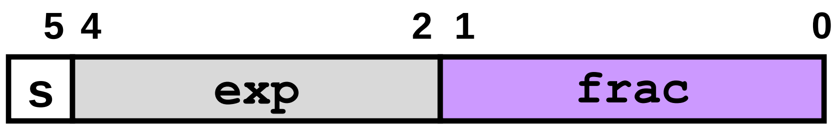
- 6-bit IEEE-like format
  - exp = 3 exponent bits
  - frac = 2 fraction bits
  - Bias is 3 (2^{3-1}-1)

- Notice how the distribution gets denser toward zero.

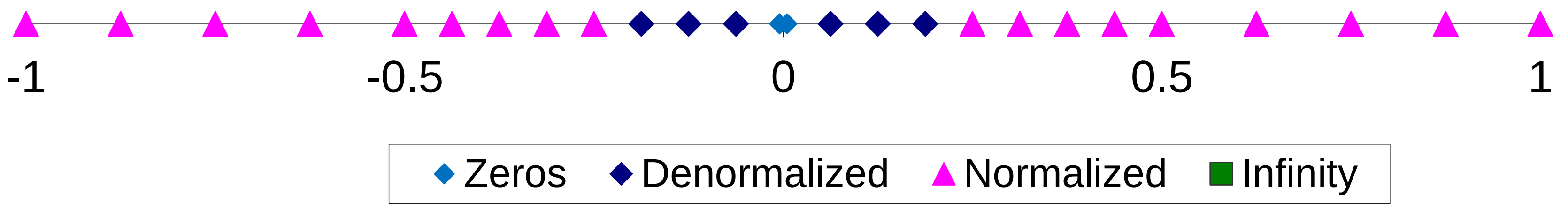


Distribution of Values (Close-up View)

- 6-bit IEEE-like format
  - exp = 3 exponent bits
  - frac = 2 fraction bits
  - Bias is 3 (2^{3-1}-1)



- Smooth transition between normalized and de-normalized numbers due to definition E = 1 - Bias for denormalized values
  - Zeros are denormalized numbers too! (+0 and -0)



Outline

- Fractional Binary Numbers
- Representing Floating Point
- Smaller Floating Point
- **Floating Point Arithmetic**

Floating Point Operations

- Conceptual view
  - $\mathtt{x\ +_{float}\ y\ =\ Fit(x\ +_{math}\ y)}$
  - $\mathtt{x\ *_{float}\ y\ =\ Fit(x\ *_{math}\ y)}$

- First compute exact, mathematical result
  - As a human: convert to decimal first, do math in decimal

- Then make it fit into desired precision
  - ***Step 1****:* Determine frac, exp
    - Frac must be of the form 1.xxxx (0.xxx if denormalized)
    - Change exp if needed to get frac to that form (e.g., if result is 101.xxx)

  - ***Step 2****:* Possibly overflow if exponent too is large
    - Unlike integer overflow, result is mathematically reasonable: infinity

  - ***Step 3****:* Possibly round to fit into frac if we have too many mantissa bits

Rounding

- Default rounding mode for IEEE floating point is Round-to-even
  - Other methods are statistically biased (round up, round down, round-to-zero)
    - Sum of set of positive numbers will consistently be over- or under- estimated
  - Round to nearest number
    - If **exactly** in between, round to nearest **even** number

- Round-to-even example
  - Illustrated with rounding of money

	\$1.40	\$1.60
Rounded	\$1	\$2

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Rounded	\$1	\$2	\$2	\$2	–\$2

Closer Look at Round-to-even

- Rounding to other decimal places than the decimal point
  - When exactly halfway between two possible values
    - Round so that least significant digit is even

  - E.g., round to nearest hundredth (i.e., 2 decimal digits in fractional part)
    - 1.23***<u>49999</u>*** => 1.23 (Less than half way)
    - 1.23***<u>50001</u>*** => 1.24 (Greater than half way)
    - 1.23***<u>50000</u>*** => 1.24 (Half way—round to even)
    - 1.24***<u>50000</u>*** => 1.24 (Half way—round to even)

Rules for IEEE754 rounding

1. Always truncate the bits that don’t fit
2. If bits were truncated, round. Two options for rounding:
    - Add 0 to least-significant remaining bit (round down)
    - Add 1 to least-significant remaining bit (round up)

- To decide which, look at the bits that are being truncated:
    - If they are less than 100…, round down (add 0)
    - If they are more than 100…, round up (add 1)
    - If they exactly equal 100…,
        - Either add 0 or 1, whichever makes the least-significant remaining bit 0
        - 0 is even

Rounding Binary Numbers

- Rules reminder:
  - If they are less than **100…**, round down (add 0)
  - If they are more than **100…**, round up (add 1)
  - If they exactly equal **100…**,
    - Either add 0 or 1, whichever makes the least-significant remaining bit 0

- Examples
  - Round to nearest 1/4 (keep 2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2+3/32	$10.00\underline{\boldsymbol{011}}_2$	10.00_2	(<1/2—down)	2
2+3/16	$10.00\underline{\boldsymbol{110}}_2$	10.01_2	(>1/2—up)	2+1/4
2+3/8	$10.01\underline{\boldsymbol{100}}_2$	10.10_2	(1/2—up to even)	2+1/2
2+5/8	$10.10\underline{\boldsymbol{100}}_2$	10.10_2	(1/2—down to even)	2+1/2
2+7/8	$10.11\underline{\boldsymbol{100}}_2$	11.00_2	(1/2—up to even)	3

Important: remember how rounding works

- Only two options when rounding
  - Leave the number alone
  - Or add one to the number

- 1010.00001001<u>10000</u>
  - Part to remove is 10…0, so we need to round
  - Options are:
    - 1010.0000100 (leave it alone)
    - 1010.0000101 (add one)

  - Pick the one that ends in zero: 1010.0000100

Mathematical Properties of FP Arithmetic

- Mathematical properties of FP Addition
  - Addition is Associative? NO
    - $(x + y) + z = x + (y + z)$
    - Possibility of overflow and inexactness of rounding
      - (3.14 + 1e10) - 1e10 = 0 (rounding)
      - 3.14 + (1e10 - 1e10) = 3.14

- Mathematical properties of FP Multiplication
  - Multiplication is Associative? NO
    - $(x \times y) \times z = x \times (y \times z)$
    - Possibility of overflow, inexactness of rounding

  - Multiplication distributes over addition? NO
    - $x \times (y + z) = (x \times y) + (x \times z)$
    - Possibility of overflow, inexactness of rounding

- More in bonus slides

Floating Point Summary

- IEEE Floating point (IEEE 754) has clear mathematical properties
  - But not always the ones you may expect!
- Represents numbers of form $(-1)^S \times M \times 2^E$
- One can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not quite the same as arithmetic on real numbers
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

Flow for floating point

- Translating between decimal and IEEE754 floating point has a clear set of steps that you always take

- Steps from Decimal to IEEE754:
  1. Decimal
  2. Floating point binary
  3. Scientific notation binary
  4. IEEE754 encoding

- Steps for IEEE754 to Decimal:
  1. IEEE754 encoding
  2. Scientific notation binary
  3. Floating point binary
  4. Decimal

Outline

- Fractional Binary Numbers
- Representing Floating Point
- Smaller Floating Point
- Floating Point Arithmetic

Outline

- Bonus slides
  - Use these for additional practice
  - And if you're interested in additional topics

Interesting Numbers for `float`/`double`

Description	exp	frac	Numeric Value{single prec., double prec.}
Zero	00…00	00…00	0.0
Smallest Pos. Denorm. • Single ~ 1.4×10^{-45} • Double ~ 4.9×10^{-324}	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
Largest Denormalized • Single ~ 1.18×10^{-38} • Double ~ 2.2×10^{-308}	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
Smallest Pos. Normalized • Just slightly larger than largest denormalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
One	01…11	00…00	1.0
Largest Normalized • Single ~ 3.4×10^{38} • Double ~ 1.8×10^{308}	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$

Normalized Encoding Example

- Value
  - **`float F = 12345.0; // single precision: k=8, n=23`**
  - $12345_{10} = 11000000111001_2 = 1.1000000111001_2 \text{ X } 2^{13}$
- Significand
  - $M = \boldsymbol{\mathit{1}}.1000000111001_2$
  - frac = 1000000111001***0000000000***
    (drop leading 1, add 10 zeros)
- Exponent
  - E = 13
  - Bias = 127
  - E = exp – Bias → exp = E + Bias = 140 =10001100_2

Floating Point Representation:

Hex:	4	6	4	0	E	4	0	0
Binary:	0100	0110	0100	0000	1110	0100	0000	0000

Creating a Floating Point Number

s	exp	frac
1	4-bits	3-bits

- Steps
  - Is the number within the range ($-2^{1\text{-Bias}}$, $+2^{1\text{-Bias}}$)?
    - If yes, “denormalize” to have a leading 0
    - otherwise, normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

- QUIZ in next three slides
  - Convert 8-bit unsigned numbers to tiny floating point format

Step 1: Normalize

s	exp	frac
1	4-bits	3-bits

- Requirement
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	*Binary*	*Fraction*	*Exponent*
128	10000000	1.0000000	7
13	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Step 2: Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- Round up conditions
  - round up if <Guard, Round, Sticky> = <x11> because >0.5
  - round up if <Guard, Round, Sticky> = <110> as per round to even rules

Value	***Fraction***	***GRS***	***Incr?***	***Rounded***
128	**1.0000000**	000	N	1.000
13	**1.1010000**	100	N	1.101
17	**1.0001000**	010	N	1.000
19	**1.0011000**	110	Y	1.010
138	**1.0001010**	011	Y	1.001
63	**1.1111100**	111	Y	10.000

Step 3: Postnormalize

- Issue
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	*Rounded*	*Exp*	*Adjusted*	*Result*
128	1.000	7		128
13	1.101	3		13
17	1.000	4		16
19	1.010	4		20
138	1.001	7		144
63	10.000	5	M=1.000 exp=6	64

Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;

float f = …;

double d = …;
```

Assume neither `d` nor `f` is NaN

```
x == (int)(double) x

x == (int)(float) x

d == (double)(float) d

f == (float)(double) f

f == -(-f);

1.0/2 == 1/2.0

d*d >= 0.0

(f+d)-f == d
```

Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;

float f = …;

double d = …;
```

Assume neither `d` nor `f` is NaN

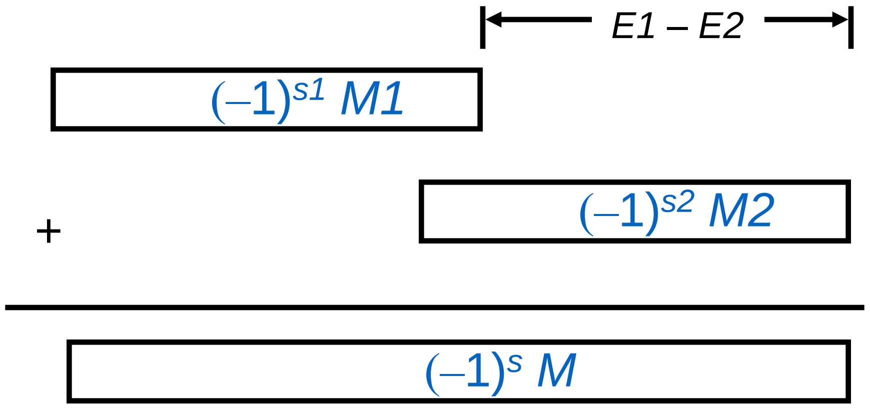
`x == (int)(double) x`	*Yes*
`x == (int)(float) x`	*No (x = TMax)*
`d == (double)(float) d`	*No (d = 1e40)*
`f == (float)(double) f`	*Yes*
`f == -(-f);`	*Yes*
`1.0/2 == 1/2.0`	*Yes*
`d*d >= 0.0`	*Yes*
`(f+d)-f == d`	*No (f = 1.0e20, d = 1.0; f+d rounded to 1.0e20*

Floating-Point Multiplication, Directly

- For cases where you can't work with exact results
  - E.g., when doing it in hardware
- Operands
  - $(-1)^{s1}\ M1\ \ 2^{E1} \quad * \quad (-1)^{s2}\ M2\ \ 2^{E2}$
- Exact result
  - $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s: s1 ^ s2
  - Significand M: M1 * M2
  - Exponent E: E1 + E2
- Fixing
  - **If M ≥ 2, shift M right, increment E**
  - If E out of range, overflow
  - Round M to fit frac precision
- Implementation
  - Biggest chore is multiplying significands

```
E1=3   M1=1.11010010
E2=5   M2=1.11001110
----------------------------------------
E=8    M=11.01001000111111
E=8+1  M=1.101001000111111
E=9    M=1.1010010010
```

Floating-Point Addition, Directly



- Operands
  - $(-1)^{s1}$ M1 2^{E1}
  - $(-1)^{s2}$ M2 2^{E2}
  - Assume $E^1 > E^2$
- Exact Result
  - $(-1)^{s}$ M 2^{E}
  - Sign s, significand M: Result of signed align & add
  - Exponent E: E^1
- Fixing
  - If M ≥ 2, shift M right, increment E
  - if M < 1, shift M left k places, decrement E by k
  - Overflow if E out of range
  - Round M to fit frac precision

E1=5 M1=1.11010010
E2=2 M2=1.11001110
E2=2 M2=0001.11001110

E1=5 M1=1.11010010
E2=5 M2=0.00111001110

E =5 M =10.00001011110
E =6 M =1.000001011110

Mathematical Properties of FP Add

- Compare to those of Abelian Group
  - Closed under addition? YES
    - But may generate infinity or NaN
  - Commutative? YES
  - Associative? NO
    - Overflow and inexactness of rounding
      - (3.14+1e10)-1e10=0 (rounding)
      - 3.14+(1e10-1e10)=3.14
  - 0 is additive identity? YES
  - Every element has additive inverse? ALMOST
    - Except for infinities & NaNs
- Monotonicity
  - $a \geq b \Rightarrow a+c \geq b+c$? ALMOST
    - Except for NaNs

Mathematical Properties of FP Multiplication

- Compare to commutative ring
  - Closed under multiplication? YES
    - But may generate infinity or NaN
  - Multiplication Commutative? YES
  - Multiplication is Associative? NO
    - Possibility of overflow, inexactness of rounding
  - 1 is multiplicative identity?YES
  - Multiplication distributes over addition? NO
    - Possibility of overflow, inexactness of rounding
- Monotonicity
  - $a \geq b$ & $c \geq 0 \Rightarrow a *c \geq b *c$? ALMOST
    - Except for NaNs