Lecture 03
Data Operations

CS213 – Intro to Computer Systems

Branden Ghena – Fall 2023

Slides adapted from:
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Northwestern

Administrivia

- You should all have access to Piazza and Gradescope
  - Contact me via email immediately if you don’t!!

- Office hours are now running
  - See Canvas homepage for office hours times

  - Mix of in-person and online hours
    - Online uses gather.town (Room C)

  - Office hours queue on the Canvas homepage

  - Note: no office hours next Monday for MLK Day

Administrivia

- Homework 1 due next week Tuesday
  - Submit on Gradescope

- Pack Lab should be out later today!
  - Sometime this evening

- Pack Lab partnership survey on Piazza
  - If you want a partner but don't know who you want to work with

- You'll do Pack Lab on one of the EECS servers
  - Usually we use Moore, but any EECS server should be fine for this lab
  - SSH + Command Line interface
  - See Piazza post with some details on accessing the servers

Today’s Goals

- Finish encodings thoughts from last time
- Explore operations we can perform on integers and more generally on binary numbers
- Understand the edge cases of those operations

Outline

- Binary and Hex

- Memory

- Encoding
- Integer Encodings
  - Signed Integers
  - Converting Sign
  - Converting Length
- **Other encodings**

Big Idea: What do bits and bytes ***mean*** in a system?

- The answer is: it depends!

- Depending on the context, the bits **`11000011`** could mean
  - The number 195
  - The number -61
  - The number -19/16
  - The character '├'
  - The **`ret`** x86 instruction

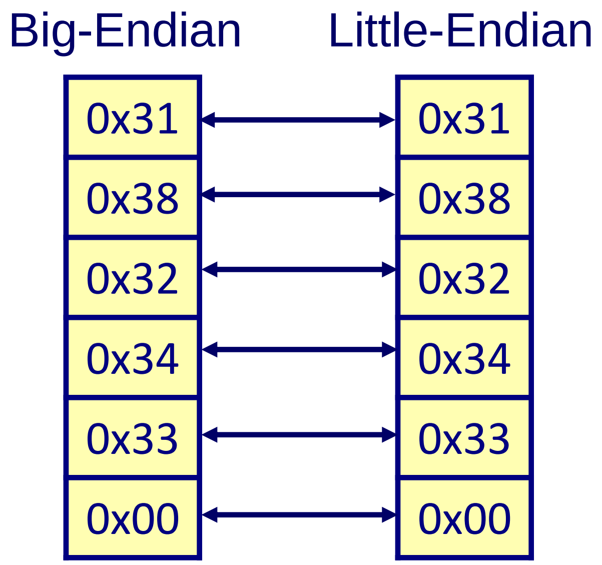
- You have to know the context to make sense of any bits you have!
  - Looking at the same bits in different contexts can lead to interesting results
  - Information = bits + context!

- An ***encoding*** is a set of rules that gives meaning to bits

Encoding strings (The C way)

- Represented by array of characters
  - Each character encoded in ASCII format
  - NULL character (code 0) to mark the end
- Compatibility
  - Byte ordering not an issue (data all single-byte!)
  - ASCII text files generally platform independent
    - Except for different conventions of line termination character(s)!

```
char S[6] = "18243";
```



Encoding color

- RGB colors
  - 3-byte values
  - First byte is Red, then Green, then Blue

- Usually specified in hexadecimal
  - #FF0000 -> maximum red, zero green or blue 🟥
  - #4E2A84 -> 1/4 red, 1/8 blue, 1/2 green (Northwestern Purple) 🟪

- 2^{24} possible colors = 16777216 colors

Interpreting file contents

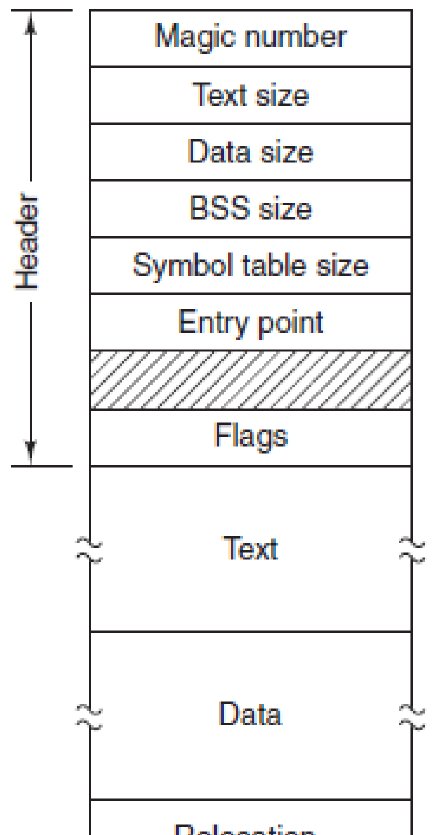
- Collections of data
  - Usually in permanent storage on your computer

- Regular files
  - Arbitrary data
  - Think of as a big array of bytes

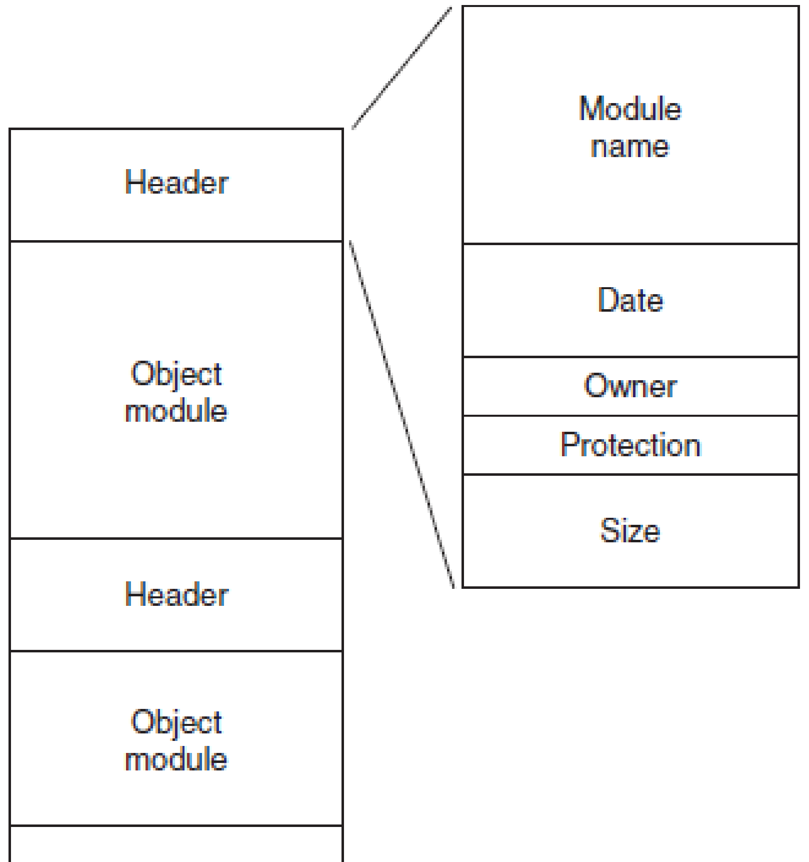
- Non-regular files would be directories, symbolic links, or other less used things

What about different types of regular files?

- Text files versus Executables versus Tar files
  - All just differing patterns of bytes!
  - It really is just all data. The meaning is in how you interpret it.



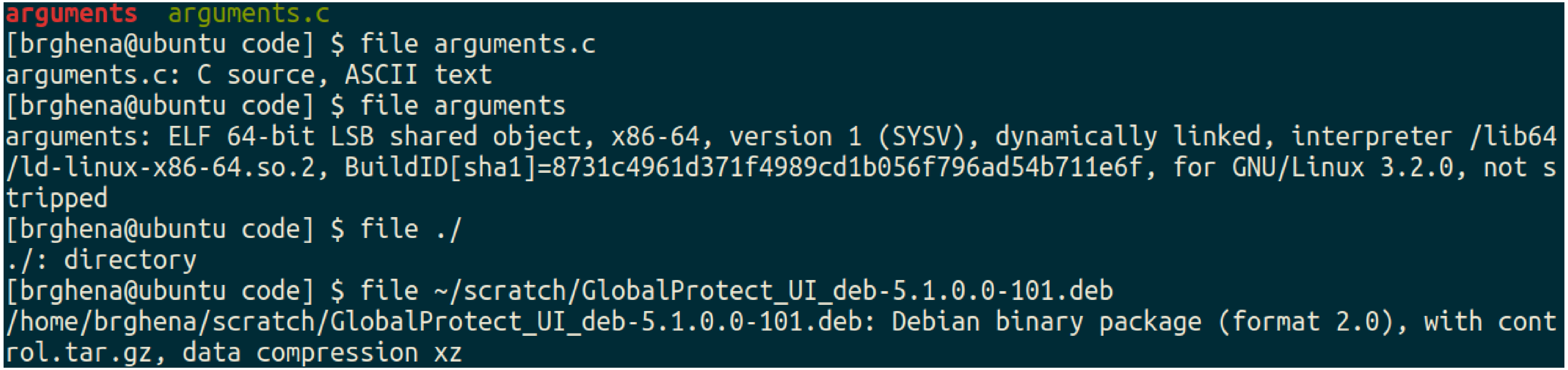
Executable File



Archive (tar)

Identifying regular files

- **`file`** in Linux command line can help determine the type of a file
  - https://github.com/file/file



```
arguments  arguments.c
[brghena@ubuntu code] $ file arguments.c
arguments.c: C source, ASCII text
[brghena@ubuntu code] $ file arguments
arguments: ELF 64-bit LSB shared object, x86-64, version 1 (SYSV), dynamically linked, interpreter /lib64
/ld-linux-x86-64.so.2, BuildID[sha1]=8731c4961d371f4989cd1b056f796ad54b711e6f, for GNU/Linux 3.2.0, not s
tripped
[brghena@ubuntu code] $ file ./
./: directory
[brghena@ubuntu code] $ file ~/scratch/GlobalProtect_UI_deb-5.1.0.0-101.deb
/home/brghena/scratch/GlobalProtect_UI_deb-5.1.0.0-101.deb: Debian binary package (format 2.0), with cont
rol.tar.gz, data compression xz
```

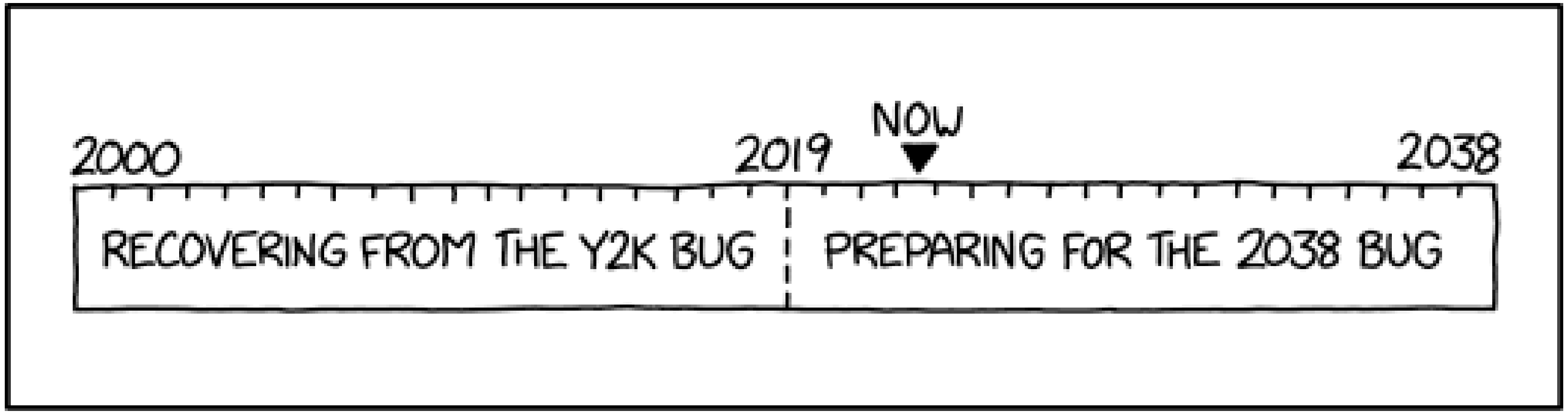
Encoding time

- Unix time:
  - 32-bit signed integer counting seconds elapsed since initial time
  - Initial time was January 1st at midnight UTC, 1970

- Current Unix time (as of last editing this slide): 1704997980
  - Negative numbers would mean times before 1970

- Problem: when does Unix time hit the maximum value?
  - 2147483647 seconds from January 1st 1970
  - Result: January 19th, 2038
  - This is the "Year 2038 Problem"

Bonus xkcd comic



REMINDER: BY NOW YOU SHOULD HAVE FINISHED YOUR Y2K RECOVERY AND BE SEVERAL YEARS INTO 2038 PREPARATION.

https://xkcd.com/2697/

Outline

- **Integer Operations**
  - **Addition**
  - Negation and Subtraction
  - Multiplication and Division

- Binary Operations
  - Boolean Algebra
  - Shifting
  - Bit Masks

C versus the hardware

- Operations you can perform on binary numbers have edge conditions
  - Usually going above or below the bit width

- If we say what happens in that scenario, it’ll be what “the hardware” (i.e., a computer) does
  - In today’s examples, pretty much every computer does the same thing

- That is not the same as what C does
  - Unclear choices are left as: **UNDEFINED BEHAVIOR** 😱
  - Which is to say, the compiler can make any choice it wants

Unsigned Addition

- Like grade-school addition, but in base 2, and ignores final carry
  - If you want, can do addition in base 10 and convert to base 2. Same result! But here we’re going to understand what the hardware is doing.
- **Example: Adding two 4-bit numbers**

$$\begin{array}{cr} & {}^{1}0\,{}^{1}1\,{}^{1}0\,1 \\ + & 0011 \\ \hline & 1000 \end{array}$$

  - **$5_{10} + 3_{10} = 8_{10}$ ✓**

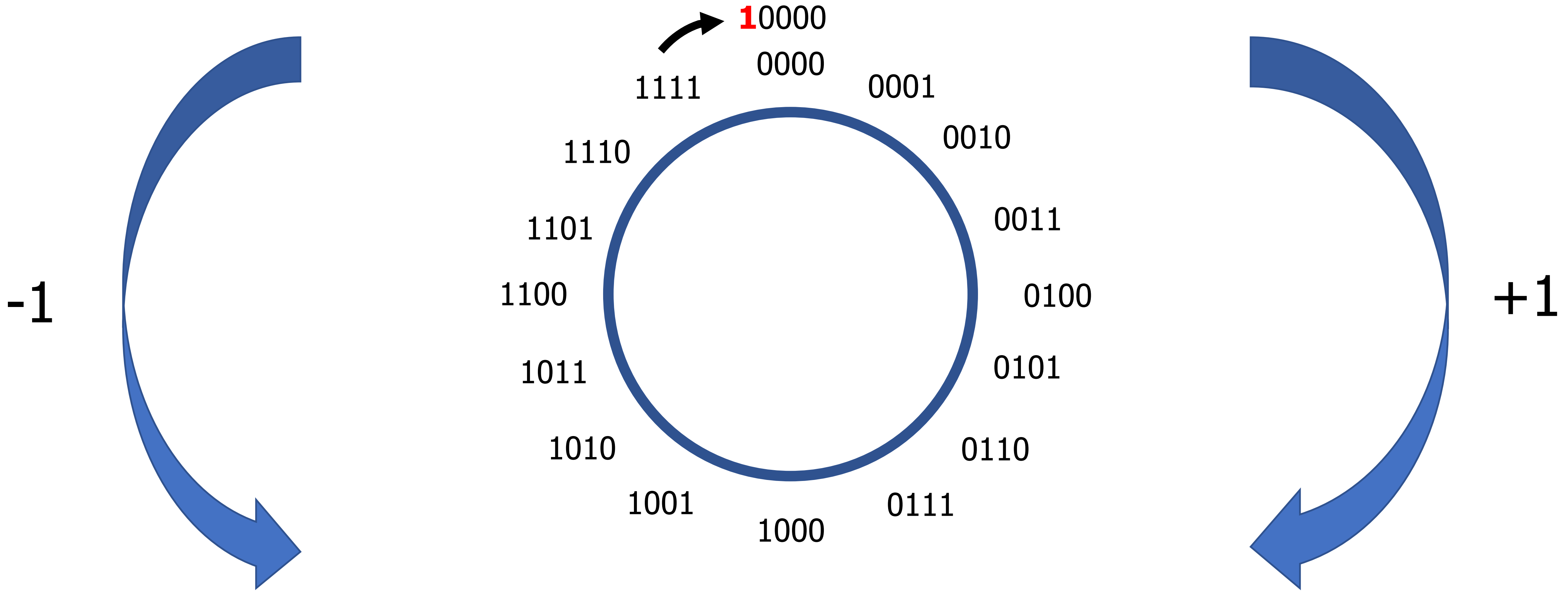
Unsigned Addition and Overflow

- What happens if the numbers get too big?
- **Example: Adding two 4-bit numbers**

```
  1 1 1 1
    1101
+   0011
  ------
   10000
```

- **13_{10} + 3_{10} = 16_{10}**
  - Too large for 4 bits! Overflow
  - Result is the 4 least significant bits (all we can fit): so 0_{10}
  - Truncate most-significant bits that do not fit
    - Gives us modular (= modulo) behavior: 16 modulo 2^4 = 0

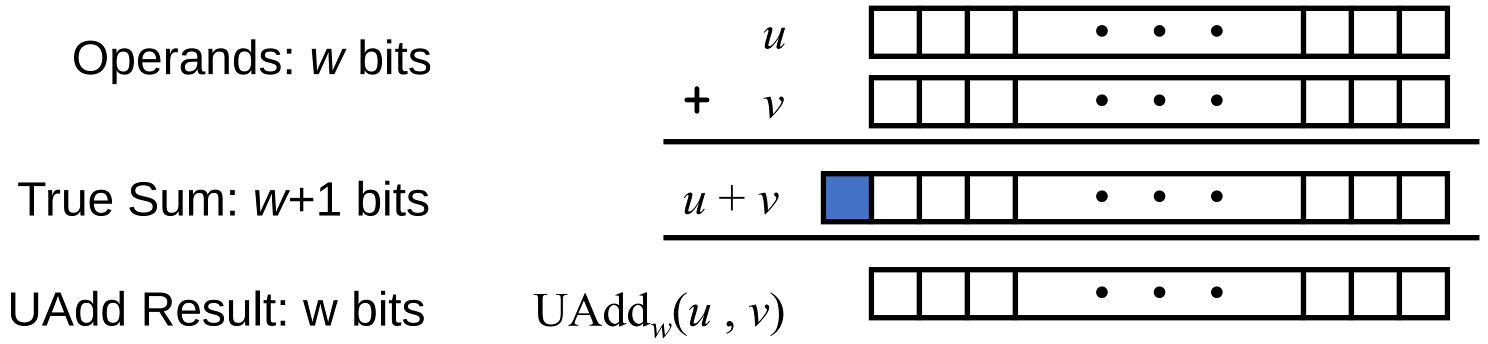
Modulo behavior in binary numbers



Unsigned addition is modular

- Implements modular arithmetic
  - $\text{UAdd}_w(u, v) = (u + v) \bmod 2^w$

- Need to drop carry bit, otherwise results will keep getting bigger
  - Example in base 10: $80_{10} + 40_{10} = 120_{10}$ (2-digit inputs become a 3-digit output!)



- Warning: C does not tell you that the result had an overflow!
  - **Unsigned** addition in C silently truncates most-significant bits beyond the limit

Signed (2’s Complement) Addition

- Works exactly the same as unsigned addition!
  - Just add the numbers in binary, and the result will work out

- Signed and unsigned sum have the exact same bit-level representation
  - Computers use the same machine instruction and the same hardware!
  - That’s a big reason 2’s complement is so nice! Shares operations with unsigned

Signed addition example

- Same addition method as unsigned
- **Example: Adding two 4-bit signed numbers**

$$
\begin{array}{rl}
 & \overset{}{1}\overset{1}{0}\overset{1}{1}1 \quad (-8 + 3 = -5) \\
+ & 0011 \quad (\qquad\qquad +3) \\
\hline
 & 1110 \quad (-8 + 6 = -2)
\end{array}
$$

- $\mathbf{-5_{10} + 3_{10} = -2_{10}}$ ✓

Combining negative and positive numbers

- Overflow sometimes makes signed addition work!
- **Example: Adding two 4-bit signed numbers**

```
 1 1 1 1
    1101   (-8 + 5 = -3)
 +  0011   (         +3)
   ------
   10000
```

- **$-3_{10} + 3_{10} = 0_{10}$**
  - Too large for 4 bits! Drop the carry bit
  - Result is what we expect as long as we truncate

Signed addition and overflow

- Overflow can still happen in signed addition though
- **Example: Adding two 4-bit signed numbers**

$$\begin{array}{r} {\scriptstyle 1\,1\,1} \\ \texttt{0101} \\ +\ \texttt{0011} \\ \hline \texttt{1000} \end{array}$$

  - $\mathbf{5_{10} + 3_{10} = -8_{10}}$ (+8 is too big to fit)
  - Remember, this was also unsigned $\mathbf{5_{10} + 3_{10} = 8_{10}}$

Signed addition and negative overflow

- Overflow also happens in the negative direction
- **Example: Adding two 4-bit signed numbers**

$$\begin{array}{cccccc}
 & {\scriptstyle 1} & & {\scriptstyle 1} & {\scriptstyle 1} & \\
 & & 1 & 0 & 1 & 1 \\
+ & & 1 & 0 & 1 & 1 \\
\hline
 & 1 & 0 & 1 & 1 & 0
\end{array}$$

  - **-5_{10} + -5_{10} = $+6_{10}$** (-10 was too small to fit)

Overflow: hardware vs C standard

- Hardware implementations for unsigned and signed addition are the same
  - Both implement truncation of overflowing bits, leads to modular arithmetic

- Unsigned overflow in C is defined as modular arithmetic

- Signed overflow in C is **UNDEFINED BEHAVIOR**
  - Compiler *probably* does modular result
  - But there are no promises about this and it can make *assumptions*
  - So don’t rely on it

Special boss in Chrono Trigger

- Dream Devourer
  - Special boss in the Nintendo DS edition

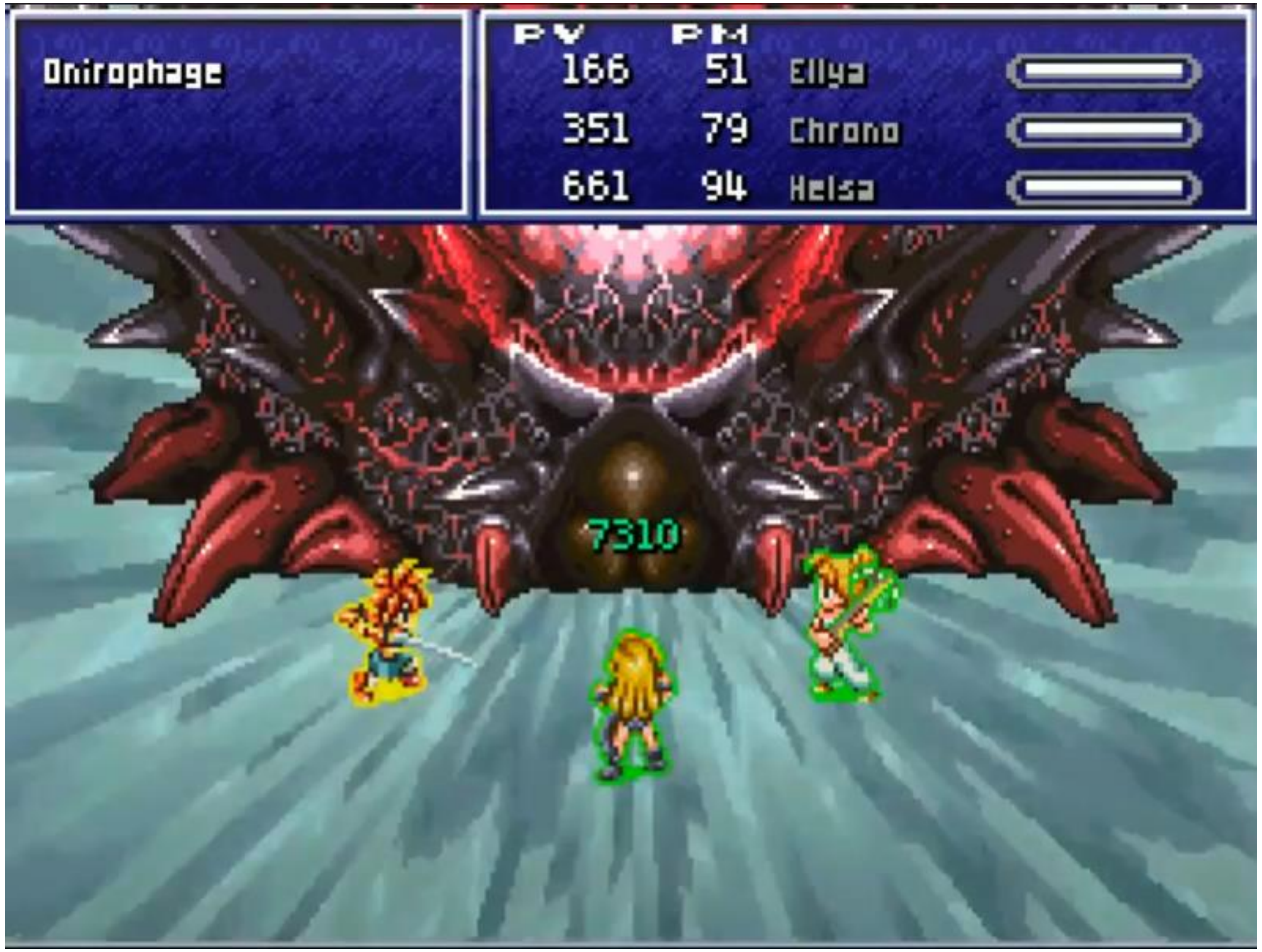
- Wanted to make it even more challenging
  - ~32000 hit points
  - Takes *forever* to defeat

- Hit points stored as a 16-bit signed integer
  - Range: -32768 to +32767

- **How do speedrunners defeat the boss?**

Chrono Trigger signed overflow bug

- Solution: heal it
- Hit points go negative and it dies



Outline

- **Integer Operations**
  - Addition
  - **Negation and Subtraction**
  - Multiplication and Division
- Binary Operations
  - Boolean Algebra
  - Shifting
  - Bit Masks

Negating a number

- In C:
  - `x = -y;`

- Operation
  - Determine the negative, signed version of the number (two’s complement)
  - Hardware method: flip bits and add one

- Completement operator (~)
  - Flips all bits: zeros become a one and ones become a zero
  - ~0b1011 -> 0b0100

Negating via Complement & Increment

- Claim: The following is true for 2's complement
  - ~x + 1 == -x

$$\begin{array}{rcccccccc} \textbf{x} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0} & \boxed{1} \\ + \quad \textbf{\textasciitilde x} & \boxed{0} & \boxed{1} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{0} \\ \hline \textbf{-1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \end{array}$$

- Complement
  - Observation: ~x + x == 1111…11$_2$ == -1

- Increment
  - ~x + 1 == ~x ***+ x - x*** + 1 == $\cancel{-1}$ - x + $\cancel{1}$ == -x

- Example, 4 bits: $6_{10} = 0110_2$
  - Complement: $1001_2 \rightarrow$ Increment = $1010_2 = -8 + 2 = -6_{10}$

Subtraction in two’s complement

- Subtraction becomes addition of the negative number
  - 5 – 3 = 5 + -3 = 2

- Both unsigned and signed subtraction
  - Convert subtractor to its two’s complement negative form
    - i.e., negate it
  - Then do addition
  - Treat result as an unsigned number

```
  11 1
   0101   (+5)
 + 1101   ( -3)
   ----
  10010
```

C rules vs hardware rules

- Exact same overflow rules apply

- Unsigned subtraction can wrap below zero to make a large number
  - Modular arithmetic

- Signed subtraction is **UNDEFINED BEHAVIOR**
  - And therefore should not be trusted

Break + practice

- Adding two 8-bit binary numbers:
  - Also determine the decimal version of the result

```
  00010101
+ 10110001
  ________
```

Break + practice

- Adding two 8-bit binary numbers:
  - Also determine the decimal version of the result

```
   11   1
  00010101
+ 10110001
  --------
  11000110
```

	Unsigned encoding		Signed encoding
00010101	16+4+1 = 21		16+4+1 = 21
10110001	128+32+16+1 = 177	OR	-128+32+16+1 = -79
11000110	128+64+4+2 = 198		-128+64+4+2 = -58

Break + practice

- Adding two 8-bit binary numbers:
  - Also determine the decimal version of the result

```
    1 1   1
  00010101
+ 10110001
  --------
  11000110
```

Unsigned encoding		Signed encoding
16+4+1 = 21		16+4+1 = 21
128+32+16+1 = 177	**OR**	-128+32+16+1 = -79
128+64+4+2 = 198		-128+64+4+2 = -58

What about unsigned subtraction 21-79?

That would treat the result as unsigned, with the value 198

Modular arithmetic in action

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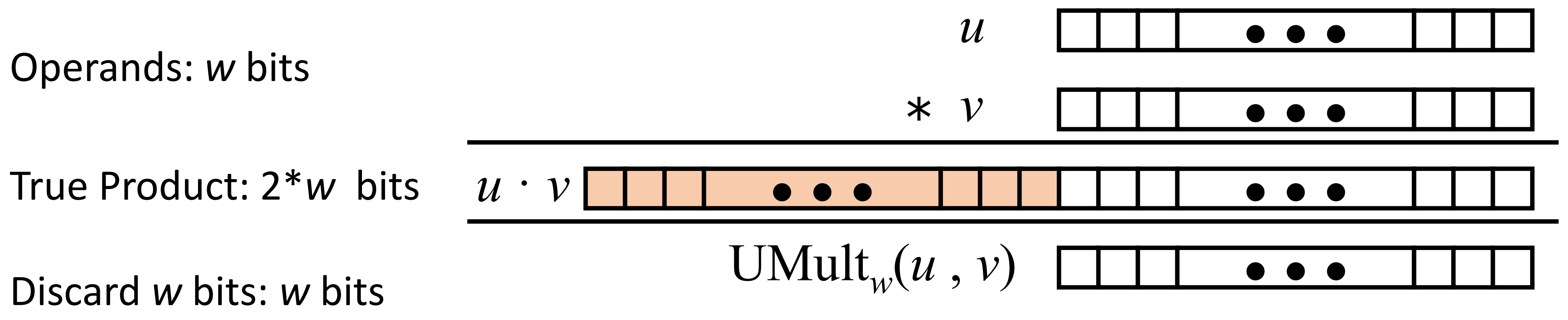
Multiplication

- Goal: Compute the Product of two ***w***-bit numbers *x*, *y*
    - Either signed or unsigned

- But, exact results can be bigger than ***w*** bits
    - Double the size (2***w***), in fact!
    - Example in base 10: $50_{10} * 20_{10} = 1000_{10}$
        - (2-digit inputs become a 4-digit output!)

- As with addition, result is truncated to fit in ***w*** bits
    - Because computers are finite, results can't grow indefinitely

Unsigned Multiplication



- **Standard Multiplication Function**
  - Equivalent to grade-school multiplication
  - But ignores most significant w bits of the result
  - As a person, we can do base 10 multiplication, convert to base 2, then truncate

- Implements modular arithmetic like addition does

  $\mathrm{UMult}_w(u, v) \;=\; (u \cdot v) \bmod 2^w$

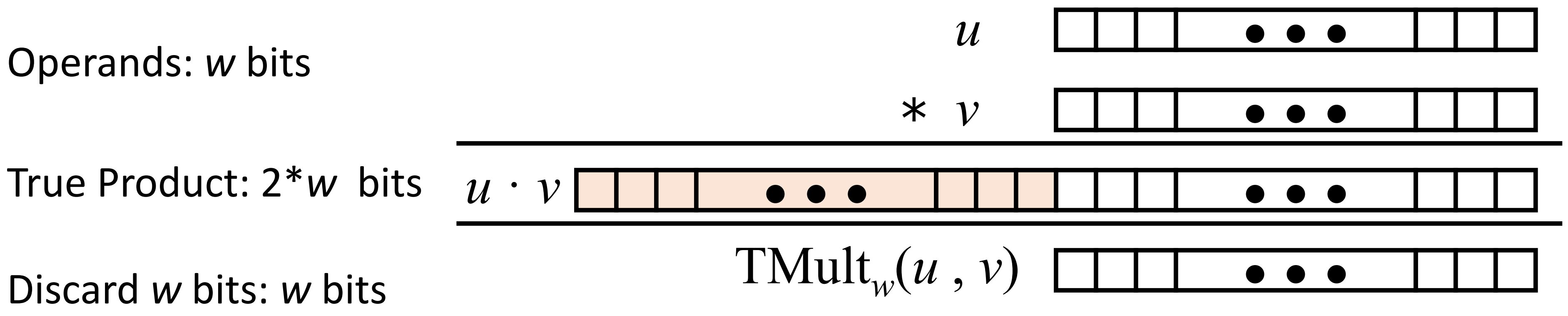
Unsigned multiplication

- **Example: Multiplying two 4-bit numbers**

$$\begin{array}{r} 0010 \\ \times\ \boxed{0}\boxed{1}\boxed{0}\boxed{1} \\ \hline 0010 \\ 00000 \\ 001000 \\ +\ 0000000 \\ \hline \cancel{000}1010 \end{array}$$

$2_{10} * 5_{10} = 10_{10}$ ✓

Signed (2’s Complement) Multiplication



- **Standard Multiplication Function**
  - Ignores most significant *w* bits
  - Lower bits still give the correct result
    - So we can use same machine instruction for both!
    - Again, that’s one reason why 2’s complement is so nice

- **In C, signed overflow is undefined**
  - ...but probably you’ll see the two’s complement behavior

Signed multiplication

- **Example: Multiplying two's complement 5-bit numbers**

```
     11110   -2
   x 00011    3
     -----
     11110
  + 111100
    ------
   1011010
```

$\cancel{10}11010$

What are these two 5-bit numbers?

What is the result of this addition?

$$\mathbf{-2_{10} * 3_{10} = -6_{10}} \ ✓$$

What about divide?

- Annoying operation, not going to discuss in this class
  - Similar to long division process
  - Tedious and complicated to get right

- I’ve worked on computers that don’t have hardware support for division at all!!

- Important thing to remember is that integers don’t have fractional parts
  - In C: `1/2 == 0`
  - We’ll need a different encoding for fractional numbers: floating point

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  - **Boolean Algebra**
  - Shifting
  - Bit Masks

Boolean algebra

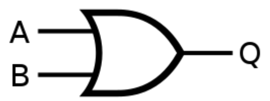
- You’ve programmed with **`and`** and **`or`** in earlier classes
  - Written **`&&`** and **`||`** in C and C++

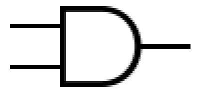
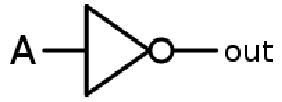
- **Boolean algebra is a generalization of that**
  - A mathematical system to represent logic (propositional logic)
  - 2 truth values: true = **`1`**, false = **`0`**
  - Operations: and **`&`**, or **`|`**, not (or complement) **`~`**

Performing Boolean algebra

- **Follow the rules for each operation to compute results**
  - Rules are like those you know from programming

  - OR: | AND: & NOT: ~ 1: True 0: False



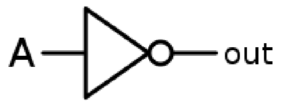



$$(1 \mid 0)\ \&\ 0 \longrightarrow 1\ \&\ 0 \longrightarrow 0$$

$$(1\ \&\ 1)\ \&\ {\sim}(0 \mid 0) \rightarrow 1\ \&\ {\sim}(0) \rightarrow 1\ \&\ 1 \longrightarrow 1$$

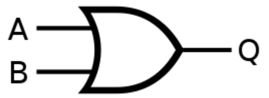
Truth tables for Boolean algebra

- For each possible value of each input, what is the output
  - Column for each input
  - Column for the output operation



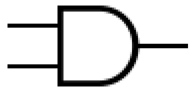
~A

A	~A
0	1
1	0



A | B

A	B	A | B
0	0	0
0	1	1
1	0	1
1	1	1



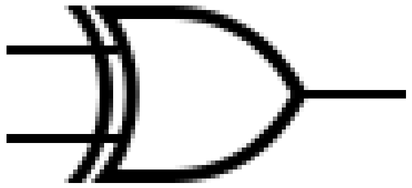
A & B

A	B	A & B
0	0	0
0	1	0
1	0	0
1	1	1

Exclusive Or (xor)

A ^ B

A	B	A ^ B
0	0	0
0	1	1
1	0	1
1	1	0



- An operation you likely haven't used before:
  - Xor - either A or B, but not both
  - ^ symbol in C
- We can build Xor out of &, |, and ~
  - A^B = (~A & B) | (A & ~B)
    - (exactly one of A and B is true)
  - A^B = (A | B) & ~(A & B)
    - (either is true but not both are true)
  - The two definitions are equivalent
    - Produce the same Truth Table

Practice problem

(A & B) | B

A	B	(A&B)|B
0	0	
0	1	
1	0	
1	1	

Practice problem

(A & B) | B

A	B	(A&B)|B
0	0	0
0	1	1
1	0	0
1	1	1

Practice problem

(A & B) | B

A	B	(A&B)|B
0	0	0
0	1	1
1	0	0
1	1	1

This is equivalent to B

(A has no influence on the solution)

Generalized Boolean algebra

- Boolean operations can be extended to work on collections of bits (i.e., bytes)

- Operations are applied one bit at a time: ***bitwise***

```
   01101001      01101001      01101001
 & 01010101    | 01010101    ^ 01010101    ~ 01010101
 ----------    ----------    ----------    ----------
   01000001      01111101      00111100      10101010
```

- All of the properties of Boolean algebra still apply
  - Relationships between operations, etc.

- Bitwise operations are usable in C: **&**, **|**, **~**, **^**
  - Can operate on any integer type (long, int, short, char, signed or unsigned)

Warning: bitwise operations are NOT logical operations

- Logical operations in C: **||**, **&&**, **!** (logical Or, And, and Not)
    - Only operate on a single bit
        - View 0 as “False”
        - View *anything nonzero* as “True”
        - Always return 0 or 1
    - Short-circuit evaluation: only checks the first operand if that is sufficient

- Examples
    - !0x41 -> 0x00 !0x00 -> 0x01 !!0x41 -> 0x01
    - 0x59 && 0x35 -> 0x01
    - (p != NULL) && *p (short circuit evaluation avoids null pointer access)

- Don’t confuse the two!! It’s a common C mistake

Break + Practice: C example of bitwise operators

```
unsigned char x = 13;
unsigned char y = 11;
unsigned char z = x & y;
```

- What decimal value is in `z` now?
    - Remember: `unsigned char` is an 8-bit value

Break + Practice: C example of bitwise operators

```
unsigned char x = 13;
unsigned char y = 11;
unsigned char z = x & y;
```

- What decimal value is in `z` now?
  - Remember: `unsigned char` is an 8-bit value

  - x: 0b00001101
  - y: 0b00001011
  - z: 0b00001001 -> 9

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  - Bit Masks

Left Shift: `x << y`

- Shift bit-vector `x` left by `y` positions
  - Throw away extra bits on left
  - Fill empty bits with 0
    - Same behavior for signed or unsigned

Argument `x`	`00000010`
`<< 3`	`~~000~~00010`*`000`*

Argument `x`	`10100010`
`<< 3`	`~~101~~00010`*`000`*

- Equivalent to multiplying by 2^y
  - And then taking modulo (i.e. truncating overflow bits)

- Undefined behavior in C when:
  - **`y < 0`**, or **`y ≥ bit_width(x)`**
  - Also when some non-0 bits get shifted off (*probably* they get truncated)

Right Shift: `x >> y`

- Shift bit-vector x right y positions
  - Throw away extra bits on right
- But how to fill the new bits that open up?
  - Will depend on signed vs unsigned
- Unsigned: Logical shift
  - Always fill with 0's on left
- Signed: Arithmetic shift
  - Replicate most significant bit on left
  - Necessary for two's complement integer representation (sign extension!)
- Undefined behavior in C when:
  - `y < 0,   or   y ≥ bit_width(x)`

Argument `x`	`01100010`
Logi. >> 2	`00011000`
Arith. >> 2	`00011000`

Argument `x`	`10100010`
Logi. >> 2	`00101000`
Arith. >> 2	`11101000`

Practice shifting in C

```
unsigned char x = 0b10100010;
  x << 3 = ?  0b00010000
```

Steps:
0b10100010**000**
0b~~101~~00010**000**

```
unsigned char x = 0b10100010;
  x >> 2 = ?   0b00101000
```

Steps:
0b**00**10100010
0b**00**101000~~10~~

```
signed   char x = 0b10100010;
  x >> 2 = ?   0b11101000
```

Steps:
0b**11**10100010
0b**11**101000~~10~~

Note:
GCC supports the prefix `0b` for binary literals (like `0x`… for hex) directly in C.
This is not part of the C standard! It may not work on other compilers.

Concept: Not all operations are equally expensive!

- Some operations are pretty simple to perform in hardware
  - E.g., addition, shifting, bitwise operations
  - Also true of doing the same by hand on paper

- Others are much more involved
  - E.g., multiplication, or even more so division
  - Consider long multiplication / long division; quite tedious!
  - Hardware is not doing the exact same thing, but similar principle

- ***Trick*: try to replace expensive operations with simple ones!**
  - Doesn’t work in all cases, but often does when mult/div by constants

Shift to divide

- Division by powers of two could be shifts
  - unsigned int x = y / 2;   unsigned int x = y >> 1;
- Even more important because division is a complicated operation
  - Multiply is implemented in (relatively) simple hardware on most systems
  - Compiler might actually translate your divide-by-powers-of-two operations into shift operations though!
- Warning: rounding needs to be handled correctly for signed numbers and division
  - See bonus slides

Compilers automatically chose the best operations

- Should you use shifts instead of multiply/divide in your C code?
  - **NO**

- Just write out the math
  - Math is more readable if that’s what you meant
  - Compiler automatically converts code to get best performance

- These two mean the same thing, but one is way more understandable
  - `int x = y * 32;`
  - `int x = (y << 5);`

Outline

- Integer Operations
    - Addition
    - Negation and Subtraction
    - Multiplication and Division

- **Binary Operations**
    - Boolean Algebra
    - Shifting
    - **Bit Masks**

Bit Masking

- How do you manipulate certain bits within a number?

- Combines some of the ideas we've already learned
  - ~, &, |, <<, >>

- Steps
  1. Create a "bit mask" which is a pattern to choose certain bits
  2. Use & or | to combine it with your number
  3. Optional: Use >> to move the bits to the least significant position

How to operate on bits

- Selecting bits, use the AND operation
  - 1 means to select that bit
  - 0 means to not select that bit

Select bottom four bits:

```
num & 0x0F
```

- Writing bits
  - Writing a one, use the OR operation
    - 1 means to write a one to that position
    - 0 is unchanged

Set 6th bit to one:

```
num | (1 << 6)
num | (0b01000000)
```

  - Writing a zero, use the AND operation
    - 0 means to write a zero to that position
    - 1 is unchanged

Clear 6th bit to zero:

```
num & (~(1 << 6))
num & (~(0b01000000))
num & (0b10111111)
```

Example: swap nibbles in byte

- Nibble - 4 bits (one hexit)
  - Input: 0x4F -> Output 0xF4

  - Method:
    - 1. Shift and select upper four bits
    - 2. Shift and select lower four bits
    - 3. Combine the two nibbles

What are the values of the new upper bits?

Unsigned -> Will be zero

```
uint8_t lower = input >> 4;
uint8_t upper = input << 4;
uint8_t output = upper | lower; // combines two halves
```

Shifting implicitly zero'd out irrelevant bits.
Otherwise we would have needed an & operation too.

Example: selecting bits

- Select bits 2 and 3 from a number

$$\begin{array}{rl} \textbf{Input:} & \texttt{0b0110}\underline{\texttt{01}}\texttt{00} \\ \textbf{Mask:} & \texttt{0b00001100} \end{array}$$

$$\begin{array}{rr} & \texttt{0b0110}\underline{\texttt{01}}\texttt{00} \\ \& & \texttt{0b00001100} \\ \hline & \texttt{0b0000}\underline{\texttt{01}}\texttt{00} \end{array}$$

Finally, shift right by two to get the values in the least significant position:

$$\texttt{0b000000}\underline{\texttt{01}}$$

In C:

```
result = (input & 0x0C) >> 2;
```

Outline

- Integer Operations
  - Addition
  - Negation and Subtraction
  - Multiplication and Division

- Binary Operations
  - Boolean Algebra
  - Shifting
  - Bit Masks

Outline

- Dividing with bit shift

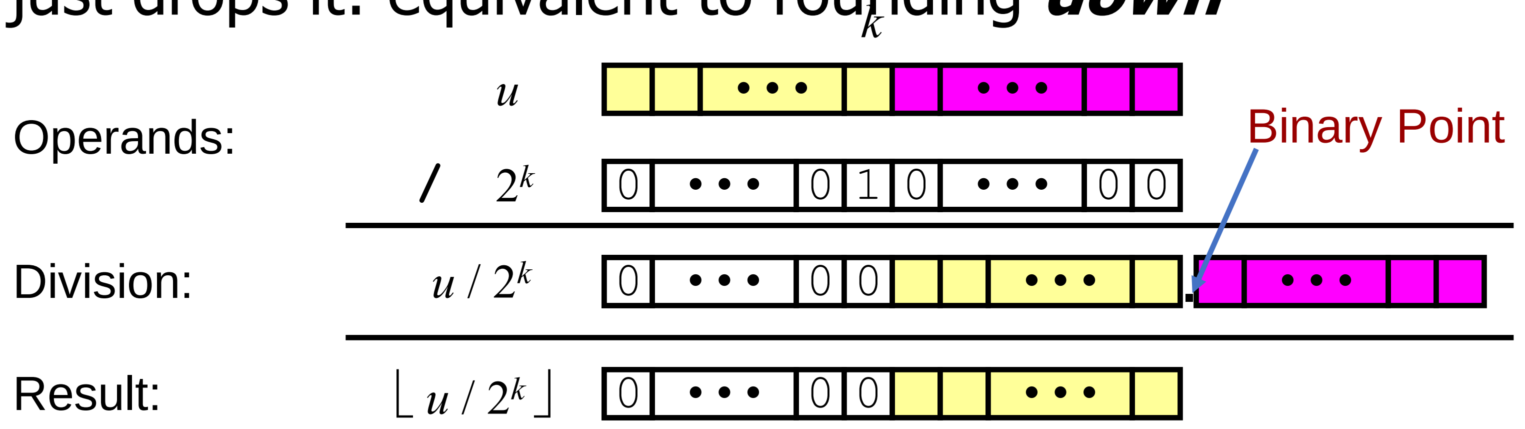
- Bonus material isn’t required and won’t be on an exam
  - Unless it becomes main lecture material in a different lecture

- Usually the material is just for students who want more depth
  - As is the case here

Unsigned Power-of-2 Divide with Right Shift

⌊ x ⌋ : round x down
⌈ x ⌉ : round x up

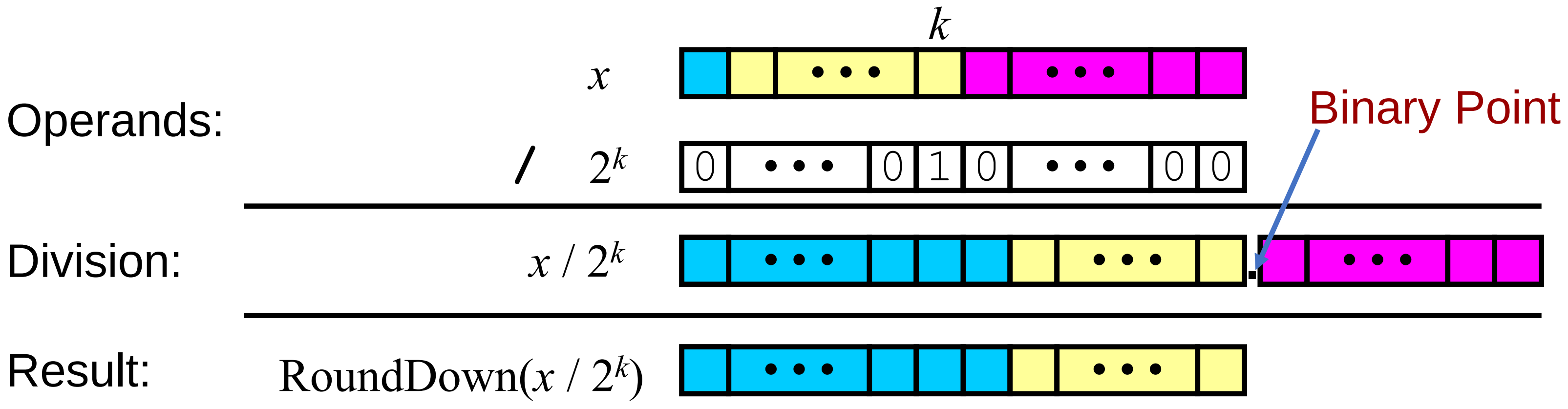
- **Quotient of unsigned by power of 2**
  - **`u >> k`** gives ⌊ **`u / 2`**^{k} ⌋
  - Uses logical shift
  - Pink part would be remainder / fractional part (right of the point)
    - Shift just drops it: equivalent to rounding ***down***



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

Signed Power-of-2 Divide with Shift (Almost)

- **Quotient of signed by power of 2**
  - **`x >> k`** gives $\lfloor \mathbf{x} / \mathbf{2^k} \rfloor$
  - Uses arithmetic shift
  - Also rounds down, again by dropping bits
    - But signed division should round ***towards 0!*** (that's its math definition)
    - That means rounding ***up*** for negative numbers!

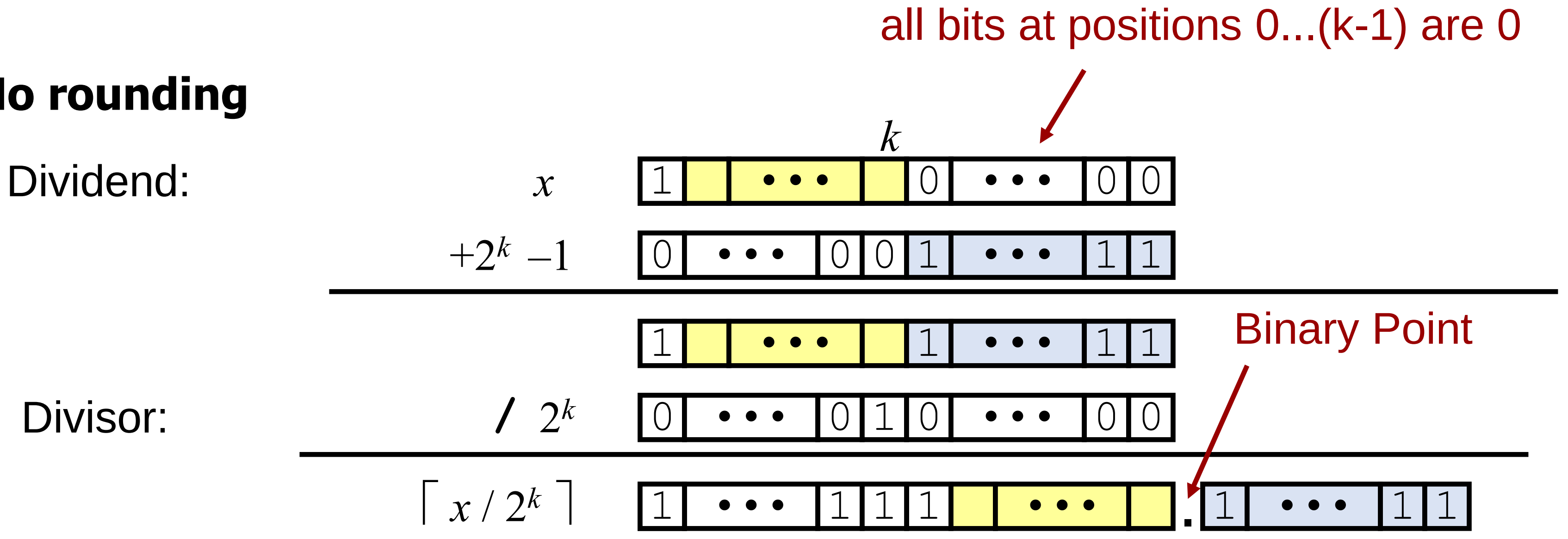


- **Example, 4 bits: -6 / 4 = -1.5 (should round towards 0, to -1)**
  - $1010_2 >> 2 = \mathit{11}10_2 = -2_{10}$
  - Rounds the wrong way!

Correct Signed Power-of-2 Divide

- Want $\lceil$ **`x / 2`**$^k\rceil$ (round towards 0)
  - Math identity: $\lceil$**x / y**$\rceil$ **=** $\lfloor$ **(x + y - 1) / y** $\rfloor$
  - Compute negative case as $\lfloor$**`(x+2`**^{k}**`–1)/ 2`**$^k\rfloor$ → gets us correct rounding!
  - Computing both cases in C: **`(x<0 ? (x + (1<<k)-1) : x) >> k`**
    - Biases dividend toward 0

- **Case 1: No rounding**

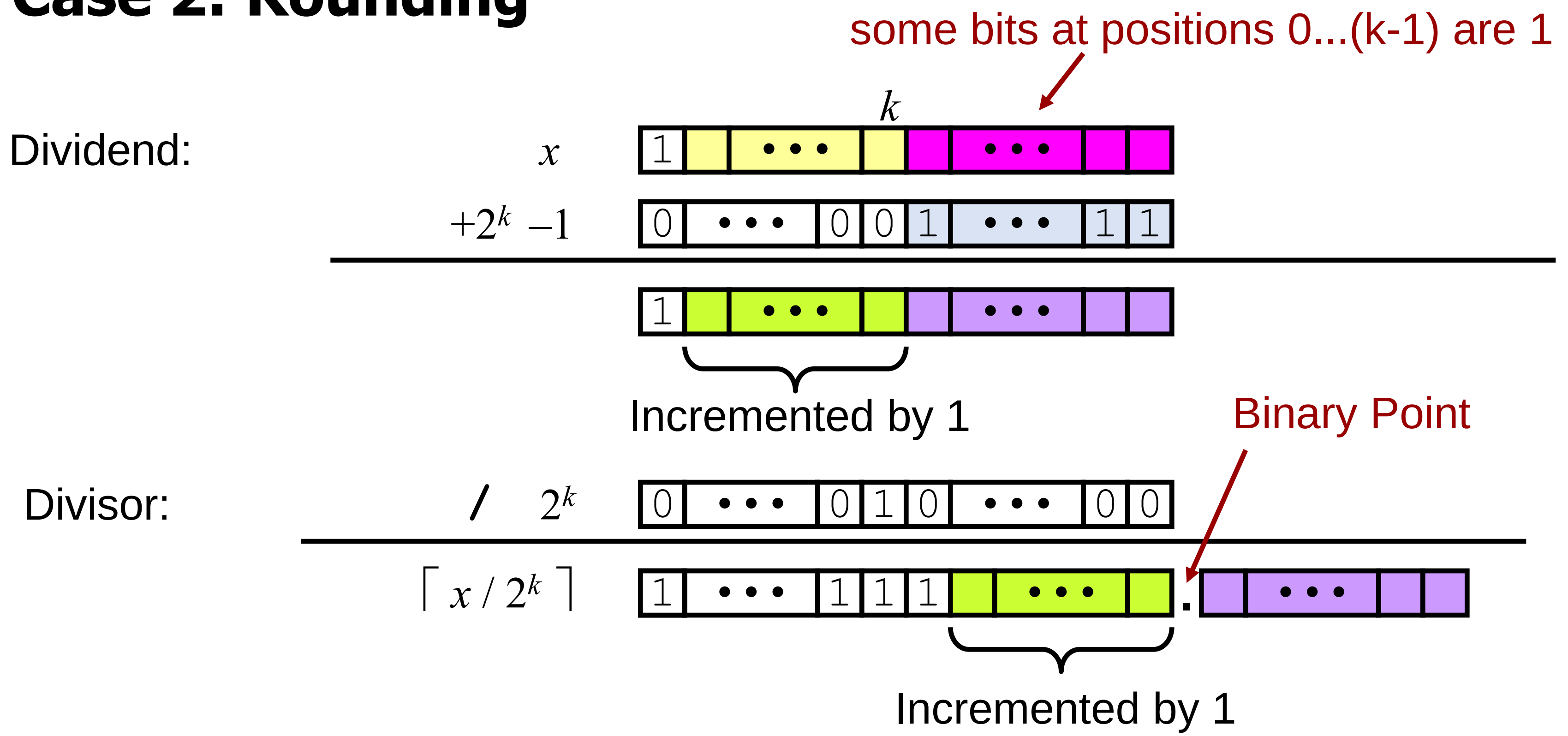


Biasing has no effect; all affected bits are dropped

- **Example, 4 bits: -8 / 2² = -2   bias = (1<<2)-1 = 3**
  - (1000 + 0011) >> 2 = 1011 >> 2 = *11*10 = -2_{10} (correct, no rounding)

Correct Signed Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result; just what we wanted

- **Example, 4 bits: -6 / 2² = -1 bias = (1<<2)-1 = 3**
  - (1010 + 0011) >> 2 = 1101 >> 2 = *11*11 = -1_{10} (correct, rounds towards 0)
- **Compiler does that for you (but you need to be able to read it!)**