Lecture 03
Integer Operations

CS213 – Intro to Computer Systems

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Slides adapted from:
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Today’s Goals

- Explore operations we can perform on binary numbers
- Understand the edge cases of those operations
- Discuss performance of various operations

C versus the hardware

- Operations you can perform on binary numbers have edge conditions
  - Usually going above or below the bit width

- If we say what happens in that scenario, it’ll be what “the hardware” (i.e., a computer) does
  - In today’s examples, pretty much every computer does the same thing

- That is not the same as what C does
  - Unclear choices are left as: **UNDEFINED BEHAVIOR** 😱

Outline

- **Addition**
- Negation and Subtraction
- Shifting
- Multiplication
- Optimizations

Unsigned Addition

- Like grade-school addition, but in base 2, and ignores final carry
  - If you want, can do addition in base 10 and convert to base 2. Same result!
- **Example: Adding two 4-bit numbers**

```
   1 1 1
   0101
 + 0011
 ------
   1000
```

  - $\mathbf{5_{10} + 3_{10} = 8_{10}}$ ✓

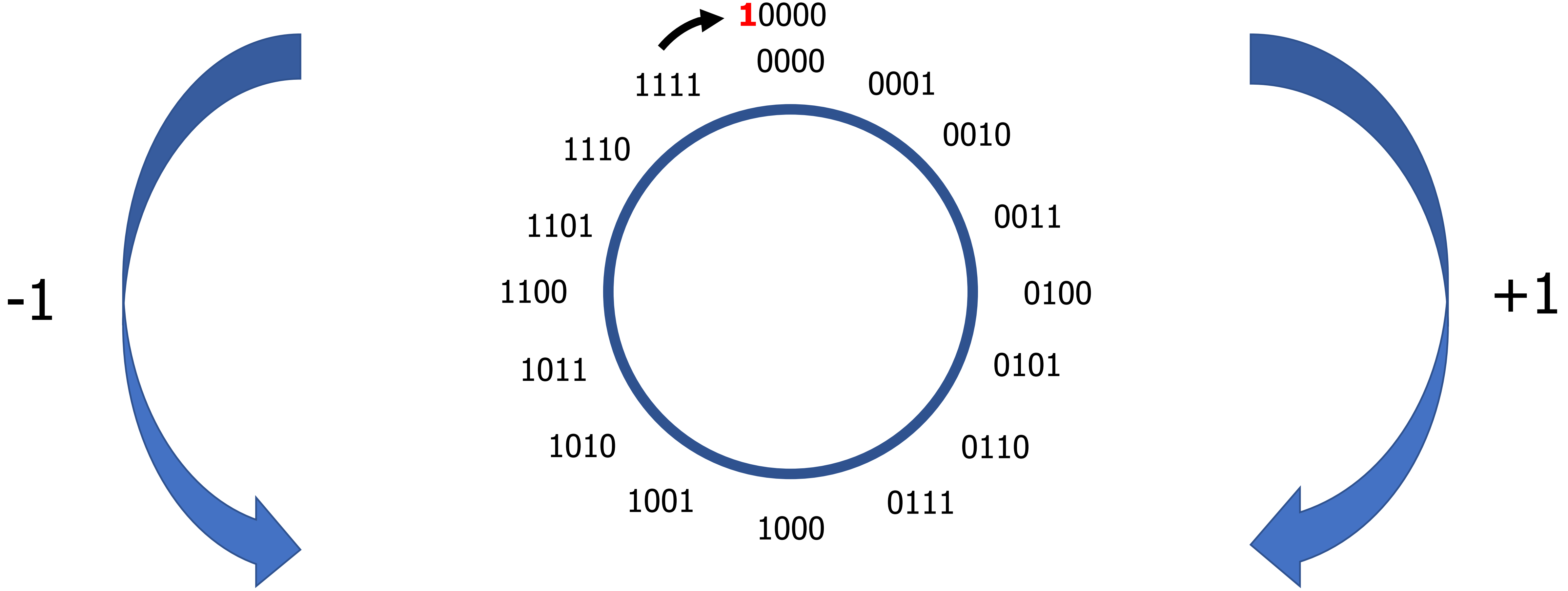
Unsigned Addition and Overflow

- What happens if the numbers get too big?
- **Example: Adding two 4-bit numbers**

$$\begin{array}{cccccc}
 & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & \\
 & & 1 & 1 & 0 & 1 \\
+ & & 0 & 0 & 1 & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 0
\end{array}$$

- **$13_{10} + 3_{10} = 16_{10}$**
  - Too large for 4 bits! Overflow
  - Result is the 4 least significant bits (all we can fit): so 0_{10}
  - Gives us modular (= modulo) behavior: 16 modulo $2^4 = 0$

Modulo behavior in binary numbers

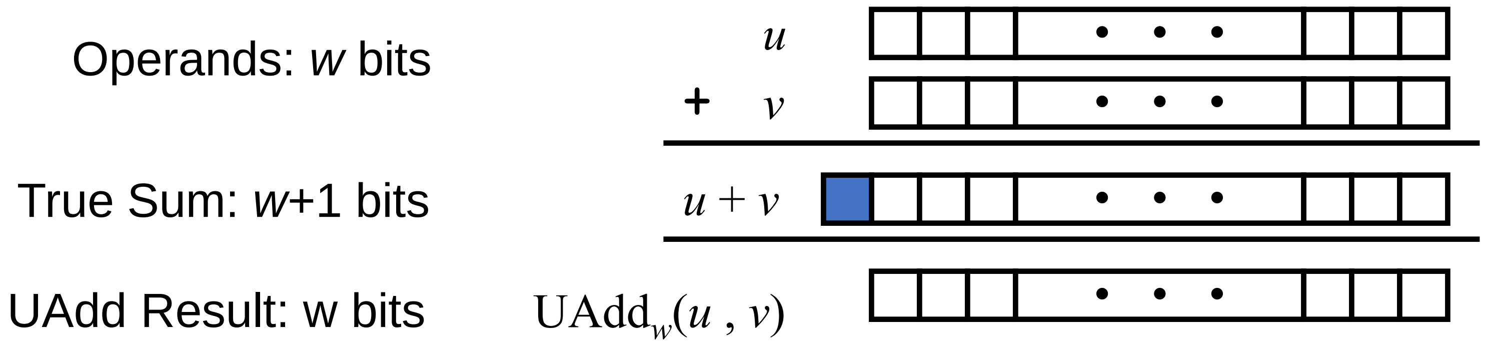


Basis for unsigned addition

$$UAdd_w(u,v) = \begin{cases} u+v & u+v<2^w \\ u+v-2^w & u+v \geq 2^w \end{cases}$$

- Implements modular arithmetic
  - s = $UAdd_w(u, v)$ = $(u + v) \bmod 2^w$

- Need to drop carry bit, otherwise results will keep getting bigger
  - Example in base 10: $80_{10} + 40_{10} = 120_{10}$ (2-digit inputs become a 3-digit output!)



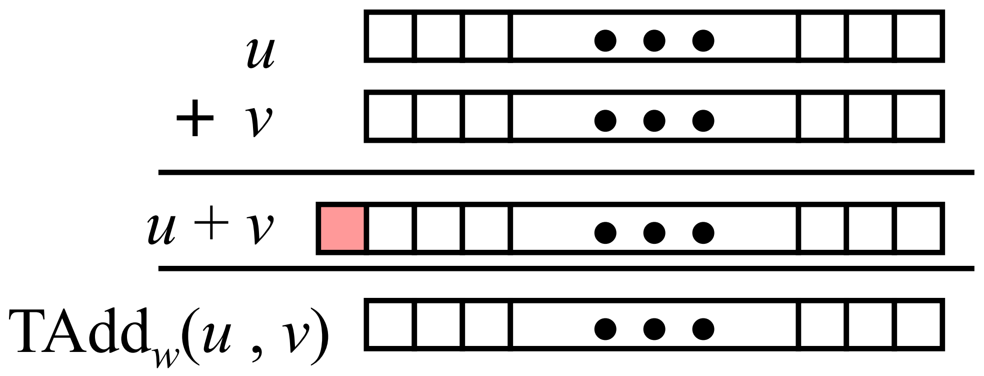
- Warning: C does not tell you that the result had an overflow!
  - Unsigned addition in C behaves like modular arithmetic

Signed (2’s Complement) Addition

Operands: *w* bits

True Sum: *w*+1 bits

TAdd Result: w bits



- TAdd and UAdd have Identical Bit-Level Behavior
  - Signed vs. unsigned addition in C:

```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

  - Will give `s == t`
- Signed and unsigned sum have the exact same bit-level representation
  - Most computers use the same machine instruction, same hardware!
  - That’s a big reason 2’s complement is so nice! Shares operations with unsigned

Signed addition example

- Same addition method as unsigned
- **Example: Adding two 4-bit signed numbers**

```
    1 1
   1011   (-8 + 3 = -5)
+  0011   (         +3)
  ------
   1110   (-8 + 6 = -2)
```

- $\mathbf{-5_{10} + 3_{10} = -2_{10}}$ ✓

Combining negative and positive numbers

- Overflow sometimes makes signed addition work!
- **Example: Adding two 4-bit signed numbers**

```
 1 1 1 1
   1101   (-8 + 5 = -3)
+  0011   (         +3)
  -----
  10000
```

- **$-3_{10} + 3_{10} = 0_{10}$**
  - Too large for 4 bits! Drop the carry bit
  - Number is what we expect

Signed addition and overflow

- Overflow can still happen in signed addition though
- **Example: Adding two 4-bit signed numbers**

```
   1 1 1
   0101
 + 0011
 ------
   1000
```

  - $\mathbf{5_{10} + 3_{10} = -8_{10}}$ (+8 is too big to fit)
  - Remember, this was also unsigned $\mathbf{5_{10} + 3_{10} = 8_{10}}$

Signed addition and underflow

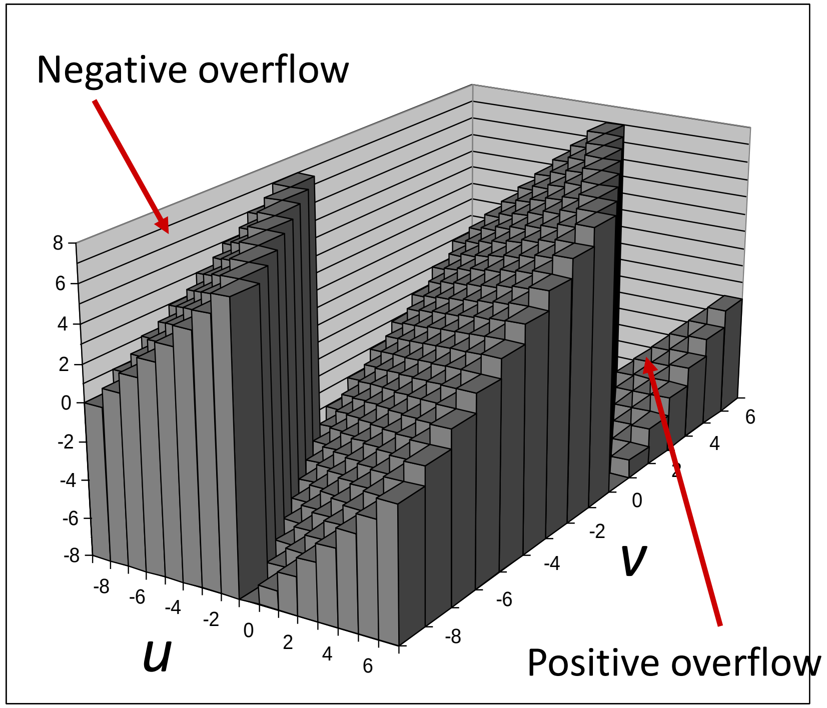
- Underflow happens in the negative direction
- **Example: Adding two 4-bit signed numbers**

$$\begin{array}{r} {}^{1}\ \ \overset{1}{1}0\overset{1}{1}1 \\ +\ \ 1011 \\ \hline 10110 \end{array}$$

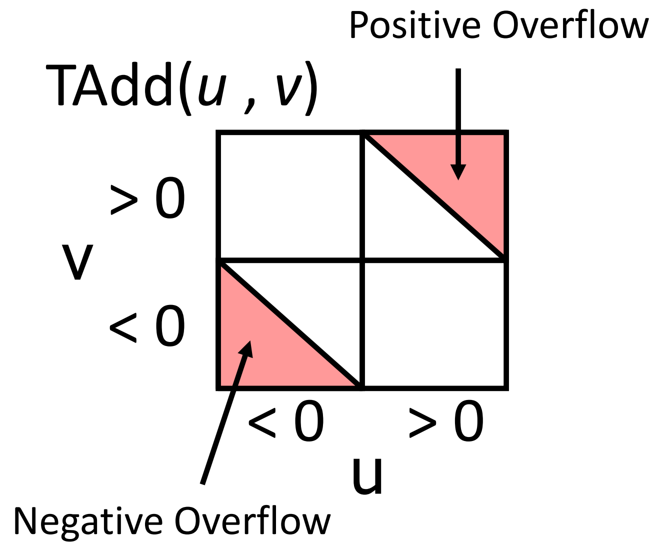
  - **-5_{10} + -5_{10} = $+6_{10}$** (-10 was too small to fit)

TAdd Overflow

- Can overflow two ways!
  - By going too far into the positives
  - *OR* too far into the negatives!
- Modular behavior either way
  - *BUT*, beware signed overflow in C
  - **UNDEFINED BEHAVIOR**
  - Compiler *probably* does modular result



$$TAdd_w(u,v) = \begin{cases} u+v+2^w & u+v < TMin_w \quad \textbf{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^w & TMax_w < u+v \quad \textbf{(PosOver)} \end{cases}$$



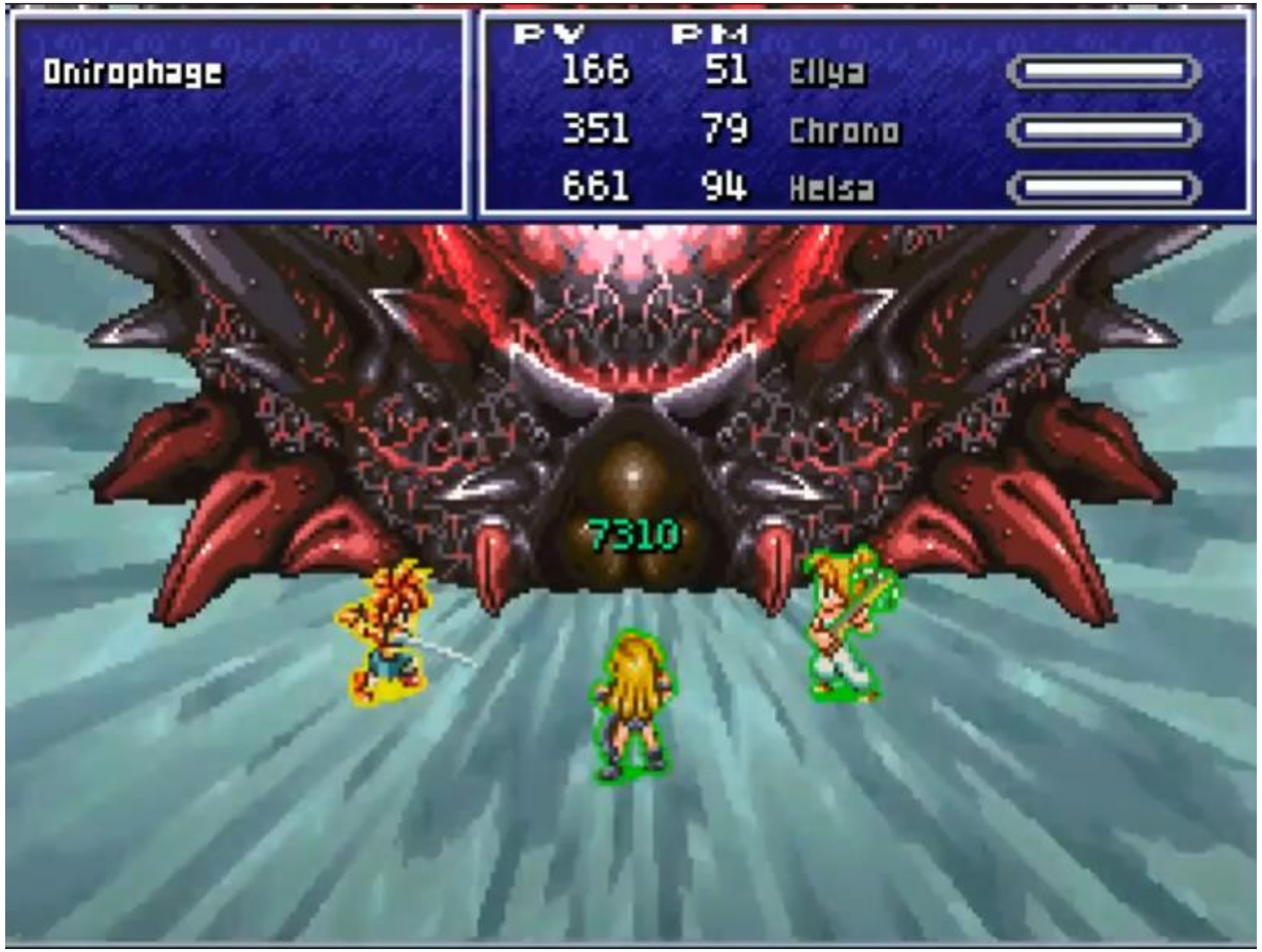
Special boss in Chrono Trigger

- Dream Devourer
  - Special boss in the Nintendo DS edition
- Wanted to make it even more challenging
  - 32000 hit points
  - Takes *forever* to defeat
- Hit points stored as a 16-bit signed integer
  - Range: -32768 to +32767

Chrono Trigger signed overflow bug

- Solution: heal it

- Hit points go negative and it dies



Outline

- Addition
- **Negation and Subtraction**
- Shifting
- Multiplication
- Optimizations

Negating with Complement & Increment

- Claim: Following holds for 2's complement
  - ~x + 1 == -x

- Complement
  - Observation: ~x + x == 1111…11$_2$ == -1

$$
\begin{array}{rr}
 & \mathbf{x}\ \boxed{1\,|\,0\,|\,0\,|\,1\,|\,1\,|\,1\,|\,0\,|\,1} \\
\mathbf{+} & \mathbf{\sim x}\ \boxed{0\,|\,1\,|\,1\,|\,0\,|\,0\,|\,0\,|\,1\,|\,0} \\
\hline
 & \mathbf{-1}\ \boxed{1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,1\,|\,1}
\end{array}
$$

- Increment
  - $\sim x + 1 == \sim x \; \boldsymbol{+ x} \; \boldsymbol{- x} + 1 == \cancel{-1} - x + \cancel{1} == -x$
  - ~x + 1 == -x

- Example, 4 bits: $6_{10} = 0110_2$
  - Complement: $1001_2 \rightarrow$ Increment $= 1010_2 = -8 + 2 = -6_{10}$

Subtraction in two's complement

- Subtraction becomes addition of the negative number
  - 5 – 3 = 5 + -3 = 2

- Unsigned subtraction
  - Treat subtractor as two's complement number and make it negative
  - Do addition
  - Treat result as an unsigned number

```
  11 1
   0101    (+5)
 + 1101    ( -3)
  -----
  10010
```

Question + Break

- In 8-bit two’s complement:
  - **What is $120_{10} – 20_{10}$?**
  - **What is 0x84 - 0x20?**

Question + Break

- In 8-bit two’s complement:
  - **What is $120_{10} – 20_{10}$?**
    - Solve as decimal. Then translate
    - $100_{10} = 01100100_2$
  - **What is 0x84 - 0x20?**
    - Solve as hexadecimal. Then translate
    - 0x64 = 0b01100100

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Left Shift: `x << y`

- Shift bit-vector x left y positions
  - Throw away extra bits on left
- Same behavior for signed and unsigned: fill open bits with 0
- Equivalent to multiplying by 2^y
  - And then taking modulo (i.e. truncating overflow bits)

Argument `x`	00000010
`<< 3`	~~000~~00010*000*

Argument `x`	10100010
`<< 3`	~~101~~00010*000*

- Undefined behavior in C when:
  - `x << y`, where `y < 0`, or `y ≥ bit_width(x)`
  - `x << y`, where some non-0 bits get shifted off (*probably* they get truncated)

Right Shift: `x >> y`

- Shift bit-vector x right y positions
  - Throw away extra bits on right
- But how to fill the new bits that open up?
  - Will depend on signed vs unsigned
- Unsigned: **Logical shift**
  - Always fill with 0's on left
- Signed: **Arithmetic shift**
  - Replicate most significant bit on left
  - Necessary for two's complement integer representation (sign extension!)
- Undefined behavior in C when:
  - `x >> y`, where `y < 0`, or `y` ≥ `bit_width(x)`

Argument `x`	`01100010`
Log. >> 2	*00*`011000`
Arith. >> 2	*00*`011000`

Argument `x`	`10100010`
Log. >> 2	*00*`101000`
Arith. >> 2	*11*`101000`

Practice shifting in C

```
unsigned char x = 0b10100010;
  x << 3 = ?    0b00010000


unsigned char x = 0b10100010; // same
  x >> 2 = ?     0b00101000


signed   char x = 0b10100010; // same
  x >> 2 = ?     0b11101000
```

Note:
GCC supports the prefix **`0b`** for binary literals (like **`0x`**… for hex) directly in C.
This is not part of the C standard! It may not work on other compilers.

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- Addition
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Multiplication

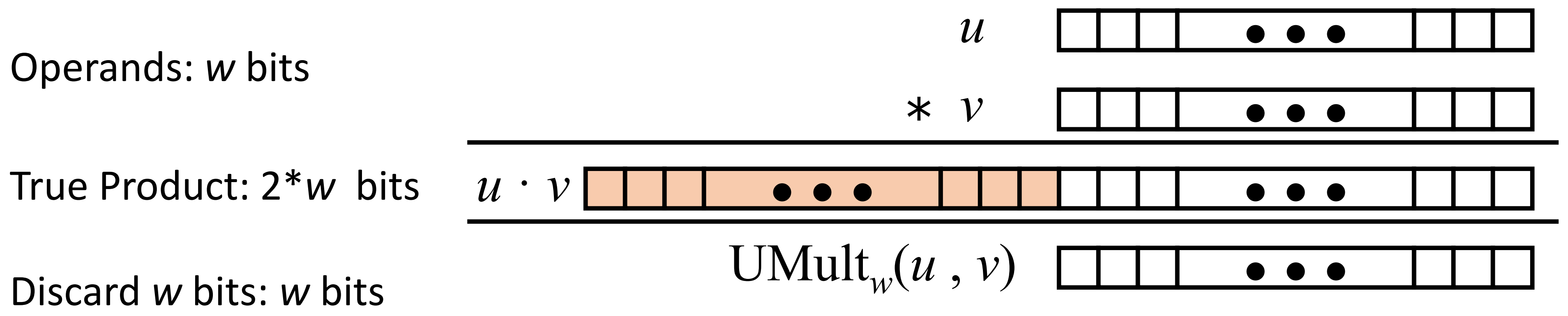
- Goal: Compute the Product of ***w***-bit numbers *x*, *y*
  - Either signed or unsigned

- But, exact results can be bigger than ***w*** bits
  - Around double the size (2*w*), in fact!

  - Example in base 10: $50_{10} * 20_{10} = 1000_{10}$
    - (2-digit inputs become a 4-digit output!)

- As with addition, result is truncated to fit in ***w*** bits
  - Because computers are finite, results can’t grow indefinitely

Unsigned Multiplication



- **Standard Multiplication Function**
  - Equivalent to grade-school multiplication
  - But ignores most significant *w* bits of the result
  - Again, can do base 10 multiplication, convert to base 2, then truncate

- **Implements Modular Arithmetic**

  $\text{UMult}_w(u, v) = \quad (u \cdot v) \bmod 2^w$

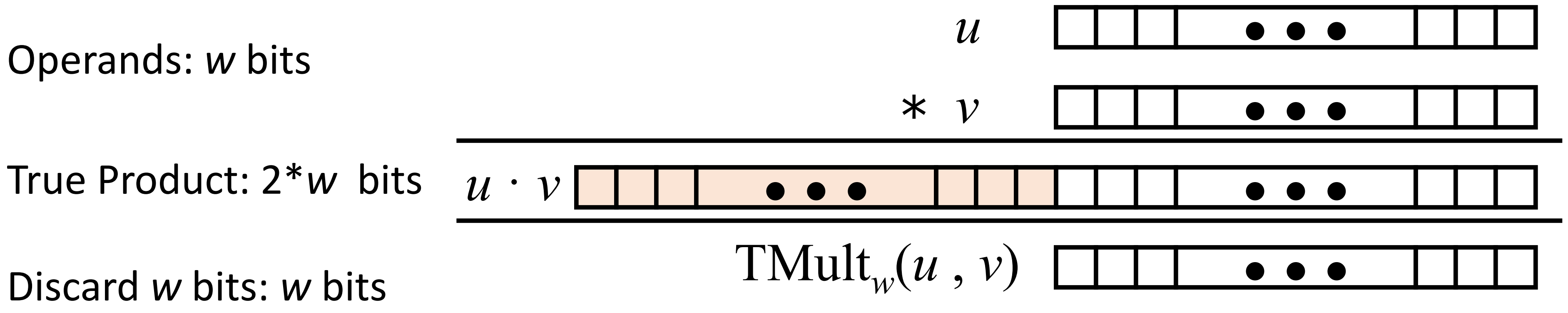
Unsigned multiplication

- **Example: Multiplying two 4-bit numbers**

$$\begin{array}{r} 0010 \\ \times\ \boxed{0}\boxed{1}\boxed{0}\boxed{1} \\ \hline 0010 \\ 00000 \\ 001000 \\ +\ 0000000 \\ \hline \cancel{000}1010 \end{array}$$

$\mathbf{2_{10} * 5_{10} = 10_{10}}$ ✓

Signed (2's Complement) Multiplication



- **Standard Multiplication Function**
  - Ignores most significant *w* bits
  - Some of which are different for signed vs. unsigned multiplication
    - Need to do sign extension to ensure correct result in upper bits
  - Lower bits are the same, so can use same machine instruction for both!
    - Again, that's one reason why 2's complement is so nice

- **In C, signed overflow is undefined**
  - ...but probably you'll see the two's complement behavior

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- **Optimizations**

What about division?

- Similar to long division process
  - Tedious and complicated to get right

- Even more complicated than multiply to make work in hardware
  - I’ve worked on a computer that didn’t even have divide

Concept: Not all operations are equally expensive!

- Some operations are pretty simple to perform in hardware
    - E.g., addition, shifting, bitwise operations
    - Also true of doing the same by hand on paper

- Others are much more involved
    - E.g., multiplication, or even more so division
    - Consider long multiplication / long division; quite tedious!
    - Hardware is not doing the exact same thing, but similar principle

- ***Trick*: ** try to replace expensive operations with simple ones!
    - Doesn’t work in all cases, but often does when mult/div by constants

Multiplication as shift operations

- Multiply 2 x 5:

```
     0010
   x 0101
   ------
     0010
    00000
   001000
+ 0000000
---------
  ~~000~~1010
```

- This is actually just bit shifts and additions

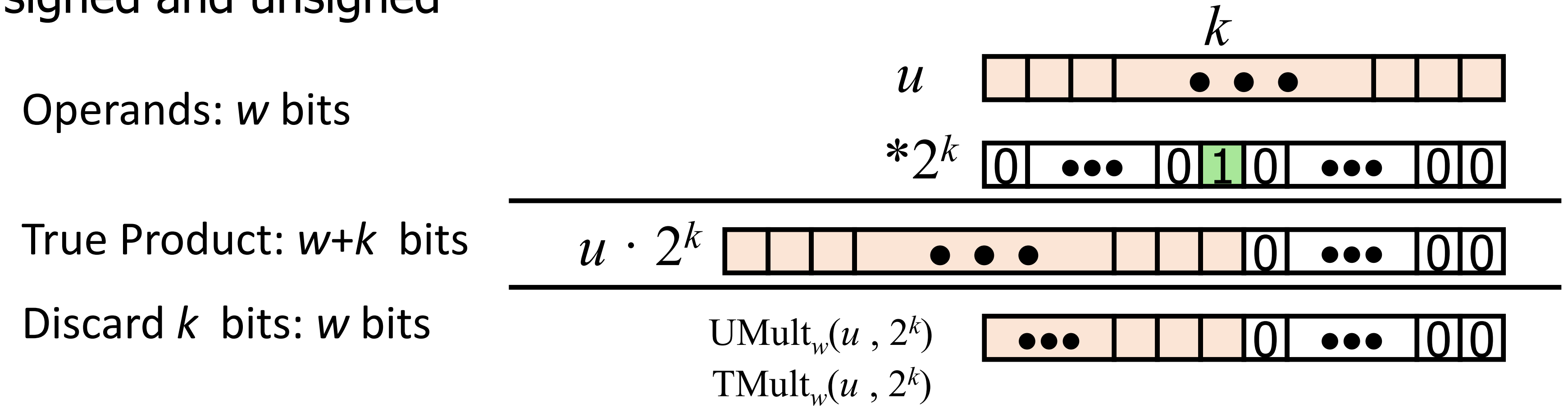
- 2 x 5 = (2 << 0) + (2 << 2)

  = 2 + 8

  = 10

Power-of-2 Multiply with Left Shift

- **Operation**
  - **`u << k`** gives **`u * 2^k`**
  - Both signed and unsigned



Operands: *w* bits
True Product: *w*+*k* bits
Discard *k* bits: *w* bits
k
u
$*2^k$
0 ••• 0 1 0 ••• 0 0
$u \cdot 2^k$
0 ••• 0 0
$\mathrm{UMult}_w(u, 2^k)$
$\mathrm{TMult}_w(u, 2^k)$

- **Examples**
  - `u << 3 ==     u * 8`
  - `(u << 5) - (u << 3) ==     u * 32 - u * 8 = u * 24`

    - Can combine multiple shifts with addition to get multiplications by non-powers-of-2

Shift to divide

- Division works too
  - unsigned int x = y / 2;   unsigned int x = y >> 1;

- Even more important because division is a complicated operation
  - Multiply is implemented in simple hardware on most systems
  - Compiler might actually translate your divide by powers of two into shift operations though!

- Warning: rounding needs to be handled correctly for signed numbers and division
  - See bonus slides

Compilers automatically chose the best operations

- Should you use shifts instead of multiply in your C code?
  - **NO**

- Just write out the multiplication
  - Multiplication is more readable if that’s what you meant
  - Compiler automatically converts code for you for best performance

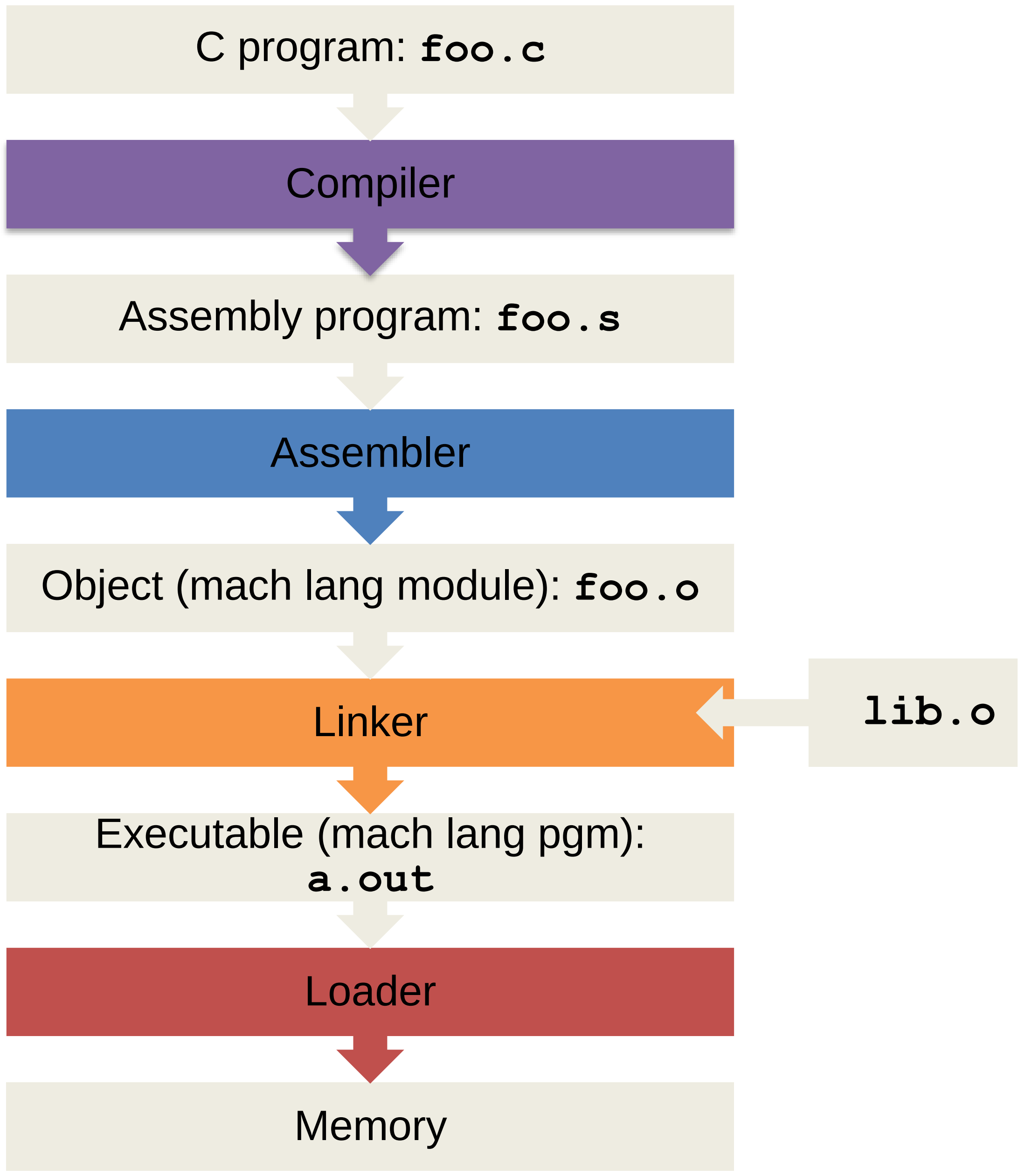
- These two mean the same thing, but one is way more understandable
    - int x = y * 32;
    - int x = (y << 5);

C code translation

- Steps for C

CALL

1. **C**ompiler
2. **A**ssembler
3. **L**inker
4. **L**oader



Compiler

- Input: higher-level language code (C, C++, Java, etc.)
- Output: assembly language code (for a particular computer)

- Process
  - Handle pre-processor (defines and includes)
  - Preform optimizations on code
    - Make it faster (such as divide-into-shift)
    - Make it use less memory (eliminate unused variables)

- Entire course worth of material here: CS322

Outline

- Addition
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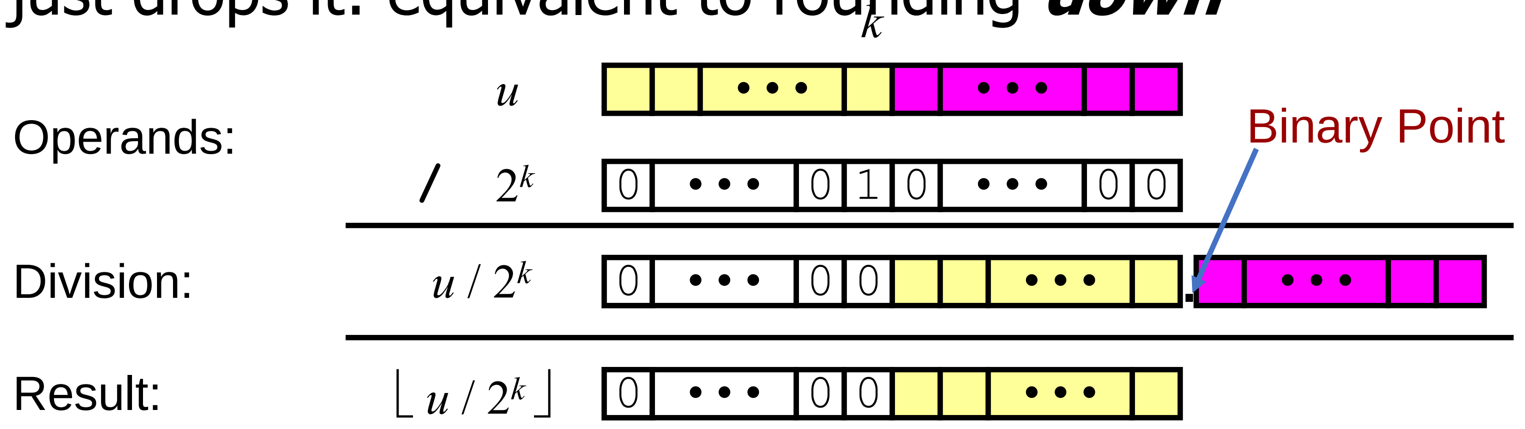
Outline

- Dividing with bit shift

Unsigned Power-of-2 Divide with Right Shift

⌊ x ⌋ : round x down
⌈ x ⌉ : round x up

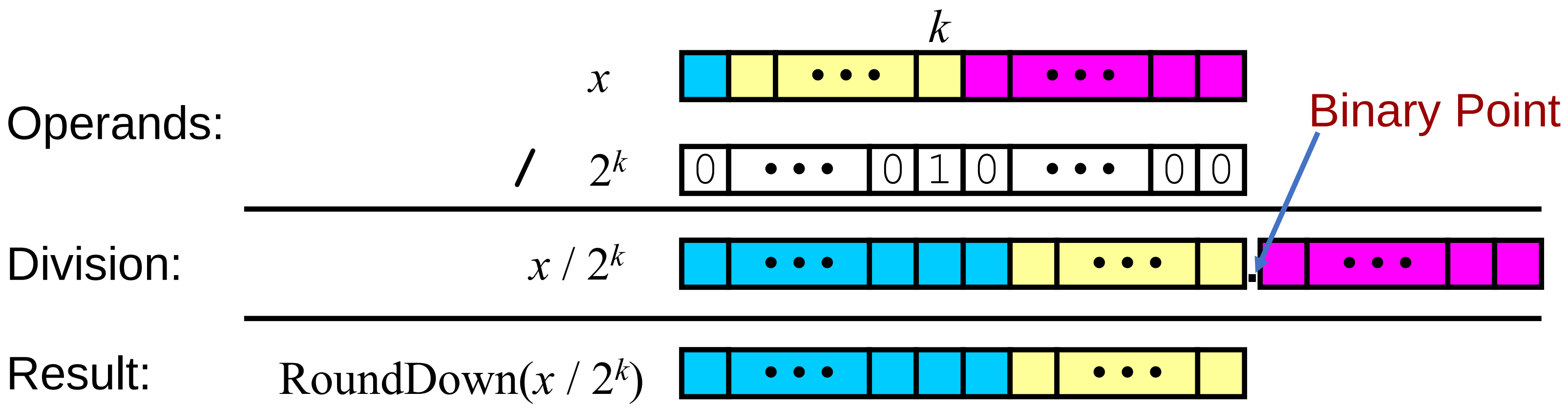
- **Quotient of unsigned by power of 2**
  - **`u >> k`** gives $\lfloor \texttt{u} / 2^k \rfloor$
  - Uses logical shift
  - Pink part would be remainder / fractional part (right of the point)
    - Shift just drops it: equivalent to rounding ***down***



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

Signed Power-of-2 Divide with Shift (Almost)

- **Quotient of signed by power of 2**
  - **`x >> k`** gives $\lfloor \mathbf{x} / \mathbf{2^k} \rfloor$
  - Uses arithmetic shift
  - Also rounds down, again by dropping bits
    - But signed division should round ***towards 0!*** (that's its math definition)
    - That means rounding ***up*** for negative numbers!

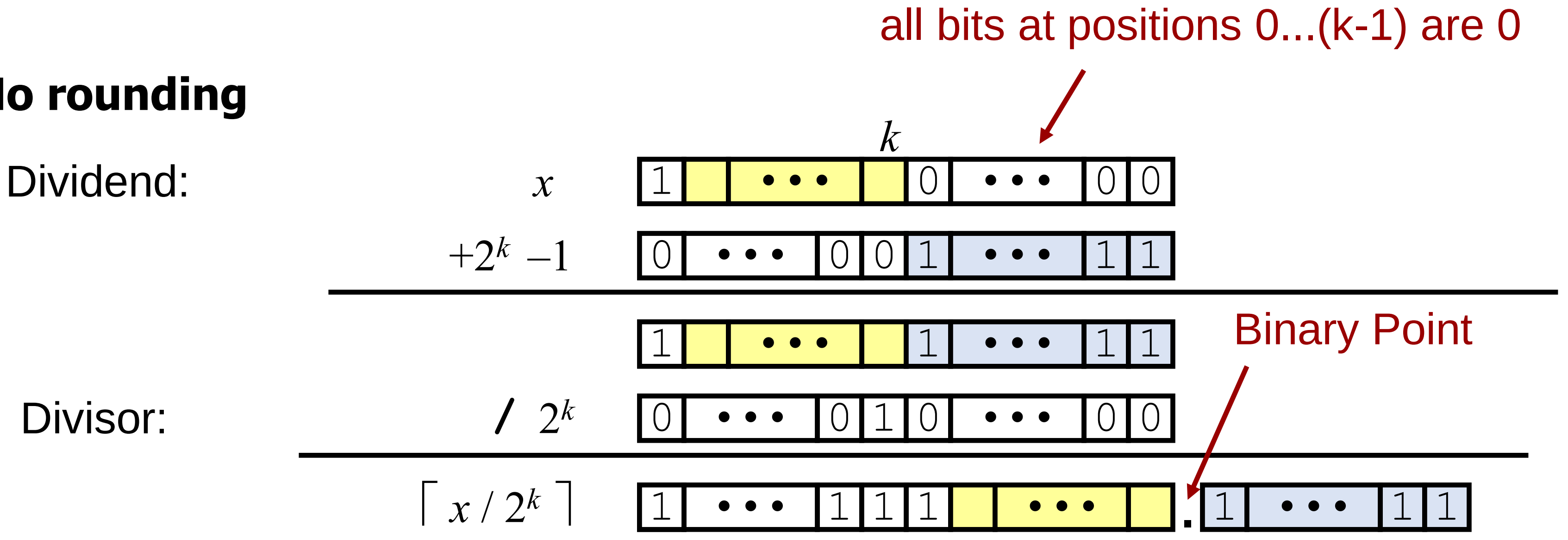


- **Example, 4 bits: -6 / 4 = -1.5 (should round towards 0, to -1)**
  - 1010_2 >> 2 = $\mathit{11}10_2$ = -2_{10}
  - Rounds the wrong way!

Correct Signed Power-of-2 Divide

- Want $\lceil$ **x / 2^k** $\rceil$ (round towards 0)
  - Math identity: $\lceil$**x / y**$\rceil$ = $\lfloor$ **(x + y - 1) / y** $\rfloor$
  - Compute negative case as $\lfloor$ **(x+2^k-1) / 2^k** $\rfloor$ → gets us correct rounding!
  - Computing both cases in C: **`(x<0 ? (x + (1<<k)-1) : x) >> k`**
    - Biases dividend toward 0

- **Case 1: No rounding**

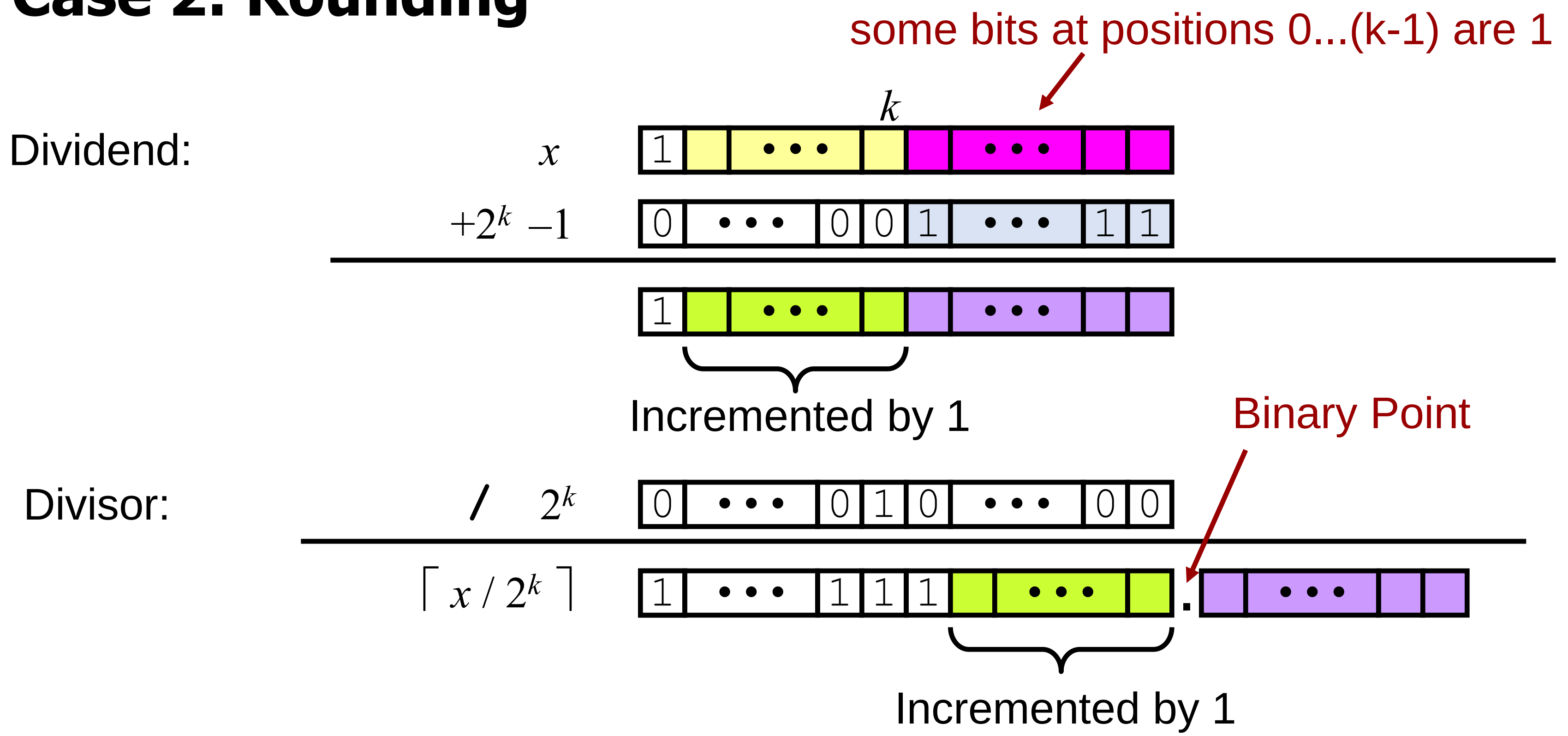


Biasing has no effect; all affected bits are dropped

- **Example, 4 bits: -8 / 2^2 = -2 bias = (1<<2)-1 = 3**
  - (1000 + 0011) >> 2 = 1011 >> 2 = *11*10 = -2$_{10}$ (correct, no rounding)

Correct Signed Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result; just what we wanted

- **Example, 4 bits: -6 / 2² = -1 bias = (1<<2)-1 = 3**
  - (1010 + 0011) >> 2 = 1101 >> 2 = *11*11 = -1_{10} (correct, rounds towards 0)
- **Compiler does that for you (but you need to be able to read it!)**