Lecture 02
Integer Representations

CS213 – Intro to Computer Systems

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Slides adapted from:
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Northwestern

Announcements

- Homework 1 is out
    - Due next week Tuesday
    - Today's lecture will finish the content you need for it

- Data Lab is available today
    - Use bit manipulations to achieve the desired goal
    - Due in two weeks
    - Everything but the floating point should be do-able after today

    - See Campuswire post about access to Moore and EECS accounts

Today's Goals

- Introduce binary operators and Boolean algebra

- Discuss data representation in memory

- Explore integer data representations
    - Signed and Unsigned numbers
    - Different bit widths
    - Translating between encoding schemes

Outline

- **Boolean Algebra**
- Data in memory
- Encoding
- Integer Encodings
  - Converting Sign
  - Converting Length

Boolean algebra

- You’ve programmed with **`and`** and **`or`** in earlier classes
  - Written **`&&`** and **`||`** in C and C++

- **Boolean algebra is a generalization of that**
  - A mathematical system to represent (propositional) logic
  - 2 truth values: true = **`1`**, false = **`0`**
  - 3 operations: and = **`&`**, or = **`|`**, not (or complement) = **`~`**

Performing Boolean algebra

- **Follow the rules for each operation to compute results**
  - Rules are the like those you know from programming
  - OR: | AND: &   NOT: ~   1: True   0: False

$$(1 \mid 0) \mathbin{\&} 0 \longrightarrow 1 \mathbin{\&} 0 \longrightarrow 0$$

$$(1 \mathbin{\&} 1) \mathbin{\&} {\sim}(0 \mid 0) \rightarrow 1 \mathbin{\&} {\sim}(0) \rightarrow 1 \mathbin{\&} 1 \longrightarrow 1$$

Truth tables for Boolean algebra

- For each possible value of each input, what is the output
  - Column for each input
  - Column for the output operation

~A

A	~A
0	1
1	0

A | B

A	B	A | B
0	0	0
0	1	1
1	0	1
1	1	1

A & B

A	B	A & B
0	0	0
0	1	0
1	0	0
1	1	1

De Morgan’s Law

- We can express Boolean operators in terms of the others

- De Morgan’s laws: swap & and |
    - A & B = ~(~A | ~B)
        - (neither A nor B is false)
    - A | B = ~(~A & ~B)
        - (A and B are not both false)
    - Useful for simplifying logical statements

Exclusive Or

A ^ B

A	B	A ^ B
0	0	0
0	1	1
1	0	1
1	1	0

- Some operations aren’t available as C logical operators
  - Xor ^ - either A or B, but not both
- We can build Xor out of &, |, and ~
  - A^B = (~A & B) | (A & ~B)
    - (exactly one of A and B is true)
  - A^B = (A | B) & ~(A & B)
    - (either is true but not both are true)
  - The two definitions are equivalent
    - Produce the same Truth Table

Generalized Boolean algebra

- Boolean operations can be extended to work on vectors of bits (i.e., bytes)

- Operations are applied one bit at a time: ***bitwise***

```
  01101001     01101001     01101001
& 01010101   | 01010101   ^ 01010101   ~ 01010101
  --------     --------     --------     --------
  01000001     01111101     00111100     10101010
```

- All of the properties of Boolean algebra still apply
  - Relationships between operations, etc.

- Bitwise operations are usable in C: **&**, **|**, **~**, **^**
  - Can operate on any integer type (long, int, short, char, signed or unsigned)

Warning: bitwise operations are NOT logical operations

- Logical operations in C: **||**, **&&**, **!** (logical Or, And, and Not)
  - Only operate on a single bit
    - View 0 as "False"
    - View *anything nonzero* as "True"
    - Always return 0 or 1
  - Short-circuit evaluation: only checks the first operand if that is sufficient

- Examples
  - !0x41 -> 0x00 !0x00 -> 0x01 !!0x41 -> 0x01
  - 0x59 && 0x35 -> 0x01
  - p && *p (short circuit avoids null pointer access)

Useful for turning many bits into 1 bit

- Don't confuse the two!! It's a common C mistake

Practice problem

(A & B) | B

A	B	(A&B)|B
0	0	
0	1	
1	0	
1	1	

Practice problem

(A & B) | B

A	B	(A&B)|B
0	0	0
0	1	1
1	0	0
1	1	1

This is equivalent to B
(A has no influence on the solution)

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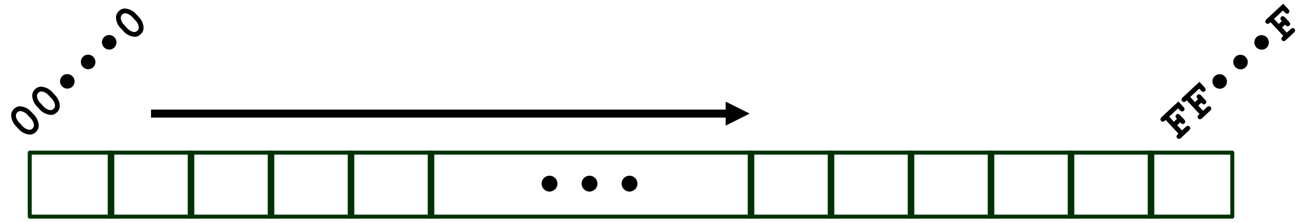
Byte-oriented memory organization

- We've seen how sequences of bits can express numbers
  - And how we usually work with groups of 8 bits (***bytes***) for convenience

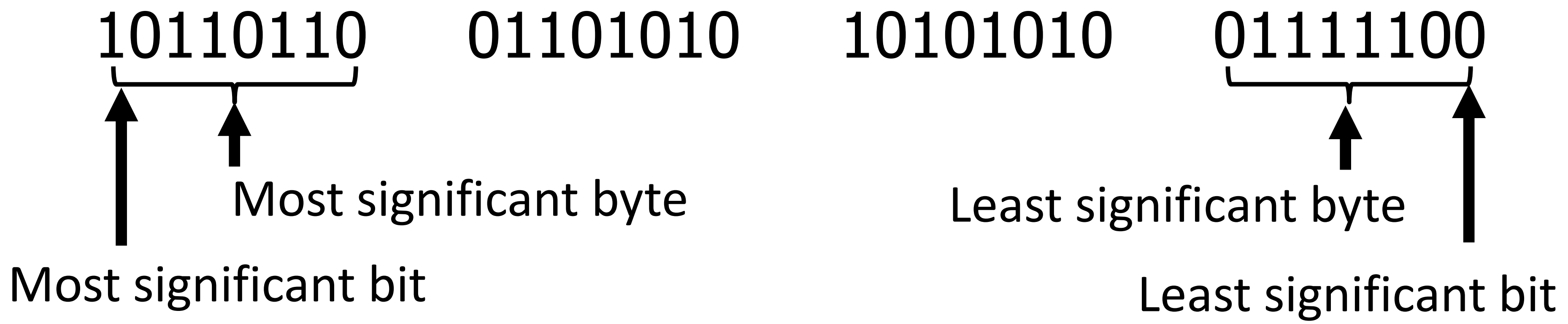
- In a computer system, bytes can be stored in memory
  - Conceptually, memory is a very large array of bytes
  - Each byte has its own address (≈ pointer)



- Compiler + run-time system control allocation
  - Where different program objects should be stored
  - Multiple mechanisms, each with its own region: static, stack, and heap

Most/least significant bits/bytes

- When working with sequences of bits (or sequences of bytes), need to be able to talk about specific bits (bytes)
  - Most Significant bit (MSb) and Most Significant Byte (MSB)
    - Have the largest possible contribution to numeric value
    - Leftmost when writing out the binary sequence
  - Least Significant bit (LSb) and Least Significant Byte (LSB)
    - Have the smallest possible contribution to numeric value
    - Rightmost when writing out the binary sequence



Addressing and byte ordering

- For data that spans multiple bytes, need to agree on two things
    - **1. What should be the address of the object?** (each byte has its own!)
        - And by extension, given an address, how do we find the relevant bytes (same question!)
    - **2. How should we order the bytes in memory?**
        - Do we put the most or least significant byte at the first address?

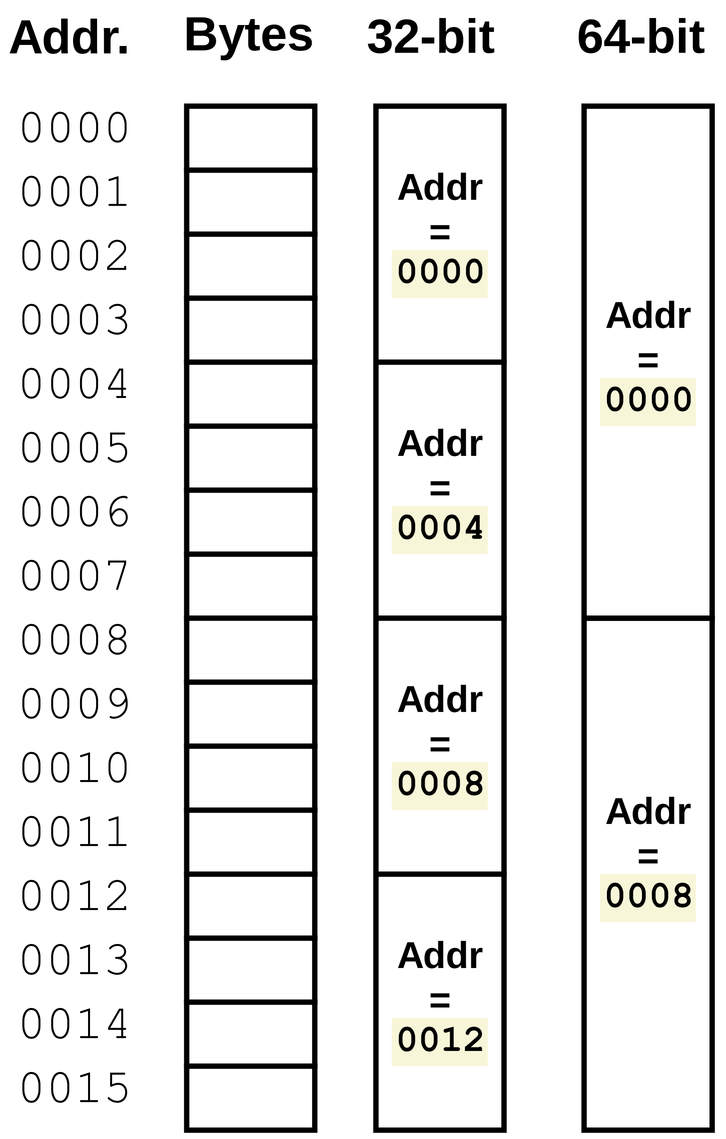
There isn’t always one correct answer

- Different systems can pick different answers! (mostly for 2nd Q)
  - Very nice illustration of two overarching principles in systems:
    You need to know the specifics of the system you’re using!
    - Many questions don’t really have right or wrong answers!
    - Instead, they have tradeoffs. What the “right” answer is depends on context!
  - Different answers across systems is perfectly fine
    - But all the parts of a given system must agree with each other!

1. Data organization in memory

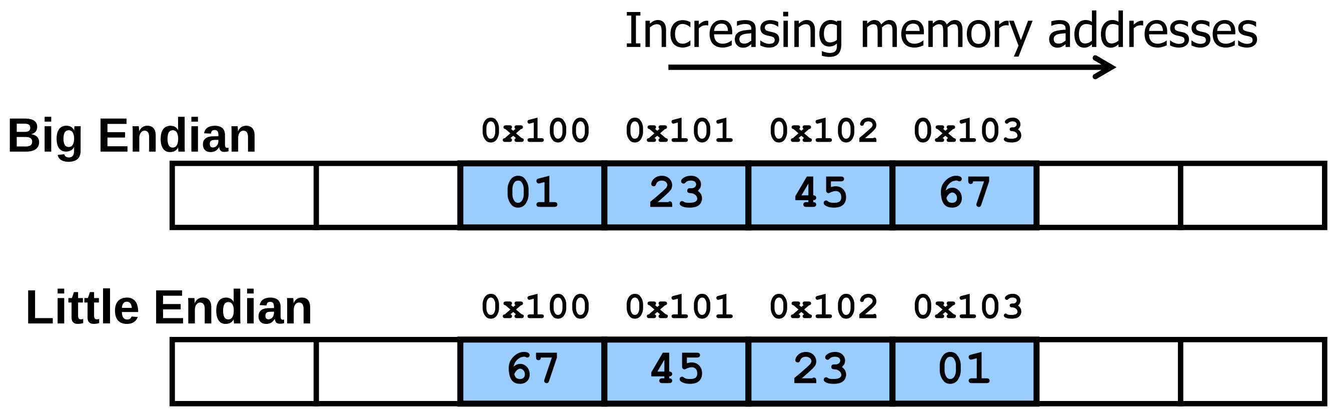
- Addresses specify byte locations
  - Address of first byte in object is used
  - Addresses of successive objects differ by 4 (32-bit) or 8 (64-bit)

- Systems pretty much universally use the address of the first byte as the address for the whole object
  - I’m not aware of any system that does otherwise
  - But there could be some weirdo systems out there (or historically)



2. Byte ordering

- How to order bytes within a multi-byte object in memory
  - Only relevant when working with data larger than a byte!
- Conventions
  - Big Endian: Oracle/Sun (SPARC), IBM (Power), computer networks
    - Most significant byte has lowest address (comes first)
  - Little Endian: Intel (x86, x86-64)
    - Least significant byte has lowest address (comes first)
- Example
  - 4-byte piece of data: **`0x01234567`**
  - Address of that data is **`0x100`**



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What do bits and bytes ***mean*** in a system?

- The answer is: it depends!

- Depending on the context, the bits **`11000011`** could mean
  - The number 195
  - The number -61
  - The number -19/16
  - The character '├'
  - The **`ret`** x86 instruction

- You have to know the context to make sense of any bits you have!
  - Looking at the same bits in different contexts can lead to interesting results
  - Information = bits + context!

- We'll see ***encodings*** that give bits meanings

Encoding characters: ASCII

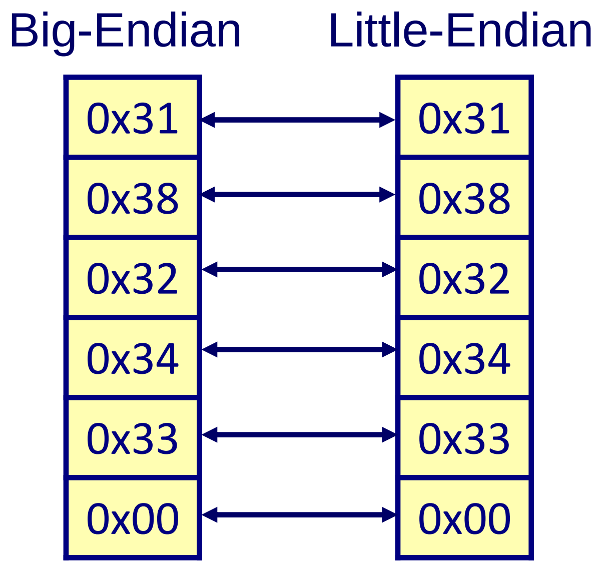
- **ASCII = American Standard Code for Information Interchange**
  - Standard dating from the 60s
- **Maps 8-bit* bit patterns to characters**
  - We already know how to go from sequences of bits (base 2) to integers
  - Need to take one more step, and interpret these integers as characters
  - (* the standard is actually 7-bit, leaving the 8th bit unused)

- Examples
  - $0100\ 0001_2 = \text{0x41} = 65_{10}$ = **`‘A’`**
  - $0100\ 0010_2 = \text{0x42} = 66_{10}$ = **`‘B’`**
  - $0011\ 0000_2 = \text{0x30} = 48_{10}$ = **`‘0’`**
  - $0011\ 0001_2 = \text{0x31} = 49_{10}$ = **`‘1’`**
- Reference: https://www.asciitable.com/

Encoding strings (The C way)

- Represented by array of characters
  - Each character encoded in ASCII format
  - NULL character (code 0) to mark the end
- Compatibility
  - Byte ordering not an issue (data all single-byte!)
  - ASCII text files generally platform independent
    - Except for different conventions of line termination character(s)!

```
char S[6] = "18243";
```



Open Question + Break

- **What things might we want to encode?**

Open Question + Break

- **What things might we want to encode?**
  - Numbers
    - Signed and unsigned integers
    - Real numbers
    - Mathematical symbols: ∞ π
  - Language
    - Characters in various different languages ΩИس서北
    - Emoji 😱😠😁😭🤷🦃🎶🎂✨🚀🍢
  - Colors, Playing Cards, User Actions, anything!

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Integer types in C

- Integer types in C come in two flavors
  - Signed: **`short`**, **`signed short`**, **`int`**, **`long`**, …
  - Unsigned: **`unsigned char`**, **`unsigned short`**, **`unsigned int`**, …

- And in multiple different sizes
  - 1 byte: **`signed char`**, **`unsigned char`**
  - 2 bytes: **`short`**, **`unsigned short`**
  - 4 bytes: **`int`**, **`unsigned int`**
  - Etc.

Sizes of C types are system dependent

- Portability
  - Some programmers assume an `int` can be used to store a pointer
  - OK for most 32-bit machines, but fails for 64-bit machines!
- How I program
  - Use fixed width integer types from <stdint.h>
  - int8_t, int16_t, int32_t
  - uint8_t, uint16_t, uint32_t

C Data Type	Intel IA32	x86-64	C Standard* (C99)
char	1	1	≥1
short	2	2	≥2
int	4	4	≥2
long	4	8	≥4
long long	8	8	≥8
float	4	4	
double	8	8	
pointer	4	8	Widths for data, code pointers may differ!

Expressing C types in bits

- Two families of encodings to express those using bits
  - ***Unsigned*** encoding for unsigned integers
  - ***Two's complement*** encoding for signed integers

- Each encoding will use a fixed size (# of bits)
  - For a given machine
  - Size + encoding family determine which C type we're representing
  - Fixed size is because computers are finite!

Unsigned integer encoding

- Just write out the number in binary
  - Works for 0 and all positive integers

- Example: encode 105_{10} as an **unsigned** 8-bit integer
  - $104_{10} = 0\times2^7 + 1\times2^6 + 1\times2^5 + 0\times2^4 + 1\times2^3 + 0\times2^2 + 0\times2^1 + 0\times2^0$

$\Rightarrow$ **`01101000`**

$\Rightarrow$ **`0x68`**

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

(Binary To Unsigned)

Bounds of unsigned integers

- For a fixed width ***w***, a limited range of integers can be expressed
  - Smallest value (we will call ***UMin***):
    - all 0s bit pattern: 000…0, value of 0
  - Largest value (we will call ***UMax***):
    - all 1s bit pattern: 111...1, value of $2^w - 1$
    - $2^w - 1 = 1\times2^{w-1} + 1\times2^{w-2} + ... + 1\times2^1 + 1\times2^0 = 11111...$
- Maximum 8-bit number = 2^8-1 = 256-1 = 255

Attempting signed encoding

- Goal: encode integers that can be positive or negative

- First attempt: we can use the most significant bit for sign
  - "Sign-and-magnitude" encoding

  - In 8-bits:
    - +4 = 00000100
    - -4 = 10000100

+127 = 01111111
-127 = 11111111

+0 = 00000000
-0 = 10000000

- Big problem: we have two representations of zero!

- Also: hardware to do math with signed and unsigned numbers gets complicated…

Two's complement encoding

- Bad news: need to make the encoding more complicated
- Good news: it will actually work

- Plan:
  - Start with unsigned encoding, but make the largest power negative
  - Example: for 8 bits, most significant bit is worth -2^7 not $+2^7$

- To encode a negative integer
  - First, set the most significant bit to 1 to start with a big negative number
  - Then, add positive powers of 2 (the other bits) to "get back" to number we want

- Example: encode -6 as a 4-bit two's complement integer
  - $-6_{10} = 1 \times -2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^1 \Rightarrow$ `0b1010` $\Rightarrow$ **`0xa`**

Two’s complement examples

- Encode -100 as an 8-bit two’s complement number
  - $-100_{10} =$ 1×-2^7 $+ 0 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$

    -128 + 0 + 0 + 16 + 8 +4 +0 +0

    Problem becomes:
    encode +28 as a 7-bit unsigned number

  - -100_{10} = 0b10011100 = 0x9C

  - **Shortcut:** determine positive version of number, flip it, and add one
    - 100_{10} = 0b01100100
    - Flipped = 0b10011011
    - Plus 1 = 0b10011100 = 0x9C   We’ll talk about binary addition next lecture

Interpreting binary signed values

- Converting binary to signed: $B2T(X) = -\underbrace{x_{w-1}}_{\text{Sign bit}} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$

- Note: most significant bit still tells us sign!! 1-> negative
  - Checking if a number is negative is just checking that top bit

- Zero problem is solved too
  - 0b00000000 = 0 0b10000000 = -128

- -1: 0b111…1 = -1 (regardless of number of bits!)

Bounds of two's complement integers

- For a fixed width ***w***, a limited range of integers can be expressed
  - Smallest value, most negative (we will call ***TMin***):
    - 1 followed by all 0s bit pattern: 100…0 = -2^{w-1}
  - Largest value, most positive (we will call ***TMax***):
    - 0 followed by all 1s bit pattern: 01...1, value of $2^{w-1} - 1$
- Beware the asymmetry! Bigger negative number than positive

Ranges for different bit amounts

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- Observations
  - $|TMin| = TMax + 1$
    - Asymmetric range
  - $UMax = 2 * TMax + 1$

- C Programming
  - #include <limits.h>
  - Declares constants, e.g.,
    - ULONG_MAX
    - LONG_MAX
    - LONG_MIN
  - Values are platform specific

Unsigned & Signed Numeric Values

X	B2U(*X*)	B2T(*X*)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- **Equivalence**
  - Same encodings for non-negative values
- **Uniqueness**
  - Every bit pattern represents unique integer value
  - Each representable integer has unique bit encoding
- $\Rightarrow$ **Can Invert Mappings**
  - Can go from bits to number and back, and vice versa
  - U2B(x) = B2U^{-1}(x)
    - Bit pattern for unsigned integer
  - T2B(x) = B2T^{-1}(x)
    - Bit pattern for two's complement integer

Practice + Break

- What range of integers can be represented with 5-bit two’s complement?

  - A -31 to +31
  - B -15 to +15
  - C 0 to +31
  - D -16 to +15
  - E -32 to +31

Practice + Break

- What range of integers can be represented with 5-bit two’s complement?

• A	-31 to +31	No asymmetry and 6-bits
• B	-15 to +15	No asymmetry
• C	0 to +31	Unsigned
[✓ • D]	[✓ -16 to +15]	Correct
• E	-32 to +31	6-bits

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Casting signed to unsigned

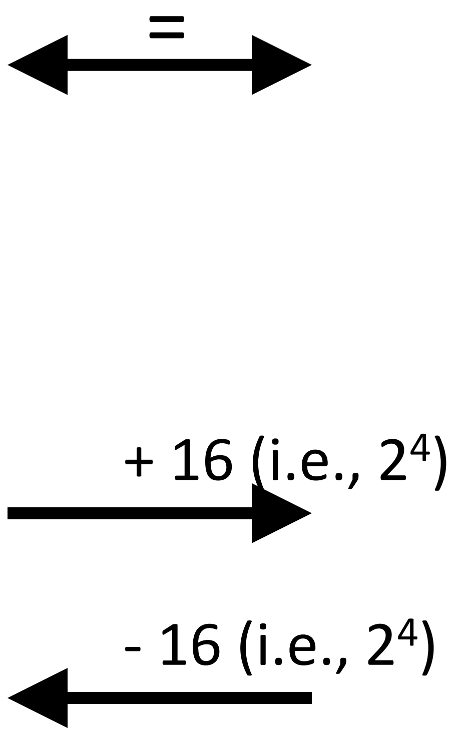
- C allows conversions from signed to unsigned (and vice versa)

```
short int          x =  15213;
unsigned short int ux = (unsigned short) x;
short int          y  = -15213;
unsigned short int uy = y; /* implicit cast! */
```

- Resulting value
  - Not based on a numeric perspective: keep the bits and ***reinterpret*** them!
  - Non-negative values unchanged
    - *ux* = 15213
  - Negative values change into (large) positive values (and vice versa)
    - *uy* = 50323
- Warning: Casts can be implicit in assignments or function calls!
  - More on that in a few slides

Mapping Signed $\leftrightarrow$ Unsigned (4 bits)

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15



Large negative factor becomes large positive!

Signed vs Unsigned in C

- Constants
  - By default are considered to be **signed integers**
  - Unsigned with "U/u" as suffix: **`0U, 4294967259U`**
- **Expression evaluation**
  - If there is a mix of unsigned and signed in a single expression, ***signed values are converted to unsigned***
  - Including comparison operations!! **<, >, ==, <=, >=**
- Can lead to surprising behavior!
  - **`-1 < 0U`** $\Rightarrow$ **false!**
  - -1 gets converted to unsigned
  - All 1s bit pattern $\Rightarrow$ UMax! Definitely not less than 0!

Code Security Example

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

- Similar to code found in FreeBSD's implementation of **`getpeername`**
- There are legions of experts trying to find vulnerabilities in programs, not all with good intentions

Typical Usage

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, MSIZE);
    printf("%s\n", mybuf);
}
```

Malicious Usage

```
/* Declaration of library function memcpy */
void *memcpy(void *dest, void *src, size_t n);
```

```
/* Kernel memory region holding user-accessible data */
#define KSIZE 1024
char kbuf[KSIZE];

/* Copy at most maxlen bytes from kernel region to user buffer */
int copy_from_kernel(void *user_dest, int maxlen) {
    /* Byte count len is minimum of buffer size and maxlen */
    int len = KSIZE < maxlen ? KSIZE : maxlen;
    memcpy(user_dest, kbuf, len);
    return len;
}
```

```
#define MSIZE 528

void getstuff() {
    char mybuf[MSIZE];
    copy_from_kernel(mybuf, -MSIZE);
    . . .
}
```

`size_t` is unsigned!

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Truncation

- May want to convert between numeric types of different sizes

- Going from a larger to a smaller number of bits is easy
  - ***Truncation***: drop bits from the most significant side until we fit
    - Values that can be represented by both types are preserved!
      - Including negative values!
    - Values that can't be represented by the smaller type are mapped to some that can (modular (= modulo) behavior)

- Example
  - 16 bits → 8 bits: ~~`10110010`~~ `01001000` → `01001000`

  - Unsigned: $45640_{10} \rightarrow 72_{10}$
    - $72_{10} = 45640_{10}$ modulo 2^8

  - Signed: $-52664_{10} \rightarrow 72_{10}$
    - $72_{10} = -52664_{10}$ modulo 2^8

Extension

- Going from smaller to larger: what to do with the “new” bits?
  - These “new” bits go on the most significant side

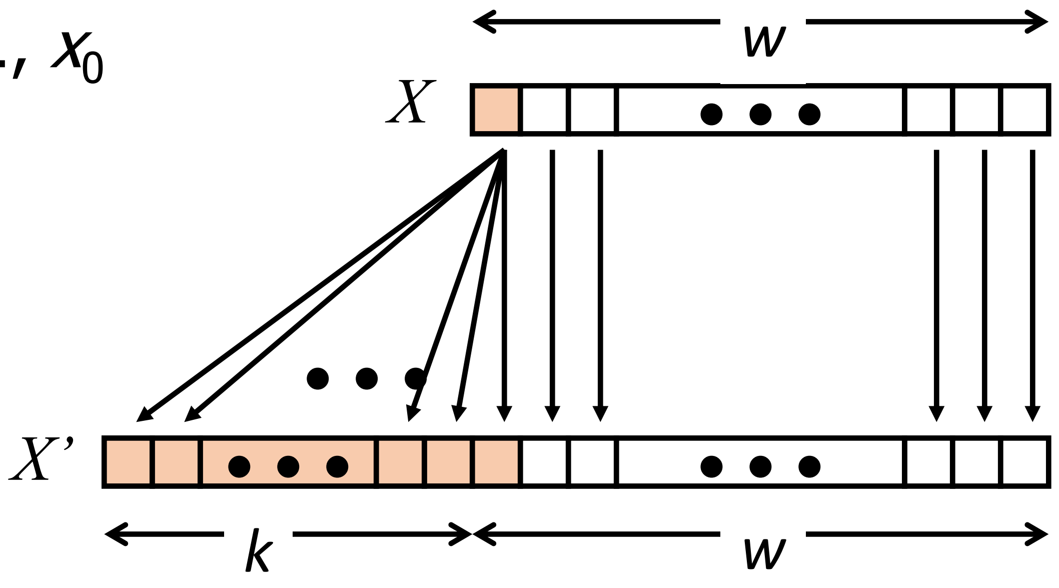
- **Unsigned**: easy, pad with 0s!
  - Always ok to add 0s on the most significant end: 15213_{10} = 00015213_{10}
  - Example: 8 bits → 16 bits: `01001000` → `00000000 01001000`
    - $72_{10} = 72_{10}$
  - Value is preserved!

- **Signed**: a bit more involved (next slides)

Sign Extension

- Task:
  - Given *w*-bit **signed** integer *x*
  - Convert it to *w*+*k*-bit integer ***with same value***
- Rule:
  - Make *k* copies of sign bit:
  - $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{}, x_{w-1}, x_{w-2}, \ldots, x_0$

k copies of MSB
(MSB = most significant bit)



Sign Extension Examples

```
short int x =  15213;
int      ix = (int) x;
short int y = -15213;
int      iy = (int) y;
```

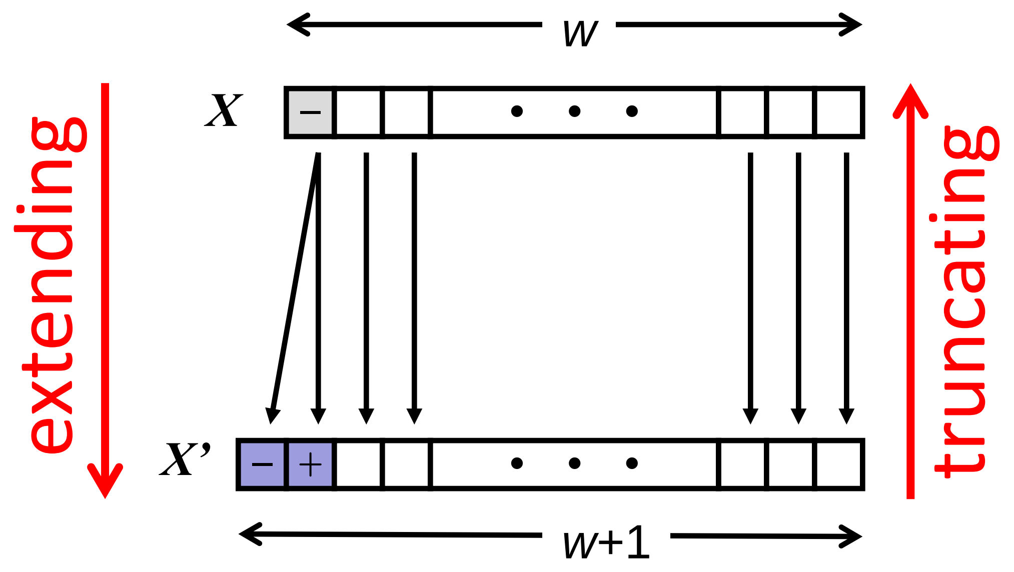
	Decimal	Hex	Binary
`x`	15213	3B 6D	00111011 01101101
`ix`	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
`y`	-15213	C4 93	11000100 10010011
`iy`	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension for signed types
  - If cast changes both sign and size, extends based on ***source*** signedness
  - But less confusing to write code that makes the types (and casts) explicit

Justification for sign extension

- Prove correctness by induction on k
  - Induction Step: extending by single bit maintains value

$$B2T(X) \quad = \quad -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$



- Look at weight of high-order bits:
  - X: $-2^{w-1}\ x_{w-1}$
  - X′: $-2^{w}\ x_{w-1} + 2^{w-1}\ x_{w-1} = (-2^{w-1+1})x_{w-1} + 2^{w-1}\ x_{w-1} = (-2\times2^{w-1} + 2^{w-1})x_{w-1} = -2^{w-1}\ x_{w-1}$

Outline

- Boolean Algebra
- Data in memory
- Encoding
- Integer Encodings
    - Converting Sign
    - Converting Length