Lecture 09
Bits, Bytes, and Integer Encoding

CS211 – Fundamentals of Computer Programming II

Branden Ghena – Fall 2021

Slides adapted from:
Jesse Tov

Northwestern

Administrivia

- Homework 4 due on Thursday
  - You can do it!
- Remember that office hours get busy right before the deadline
  - It’ll be harder to get help and you’ll get less time

Administrivia

- No lecture on Thursday
    - Take a nap instead so you can recharge

- Next week starts C++ 🎉

Today's Goals

- Go below the level of C and understand how the computer thinks about data with bits and bytes
- Understand how this leads to the boundaries of common C types
- Note: this isn't going to be on a quiz or in a homework
  - I just wanted to take today to explain more deeply
  - This will all come up again if you take CS213

Outline

- **Bits and Bytes**
- Integer Encoding
- C Type Bounds

Positional Numbering Systems

- The position of a *numeral* (e.g., digit) determines its contribution to the overall number
  - Makes arithmetic simple (compared to, say, roman numerals)
  - Any number has one canonical representation

- Example: base 10
  - $10456_{10} = 1*10^4 + 0*10^3 + 4*10^2 + 5*10^1 + 6*10^0$
  - Usually, we leave out the zeros:
    - $1*10^4 + 4*10^2 + 5*10^1 + 6*10^0$

Positional Numbering Systems

- Other bases are also possible
  - Base 60, used by the Babylonians
    - The source of 60 seconds in a minute, 60 minutes in an hour
    - And 360 degrees in a circle

  - Base 20, used by the Maya and Gauls (bits remain in French today)

  - Base 2: $10010010_2 = 1*2^7 + 1*2^4 + 1*2^1 = 146_{10}$

Base 2 Example

- Computer Scientists use base 2 a ***LOT***
- Let's convert 134_{10} to base 2
- We need to decompose 134_{10} into a sum of powers of 2
  - Start with the largest power of 2 that is smaller or equal to 134_{10}
  - Subtract it, then repeat the process

$$134_{10} - \boxed{128_{10}} = 6_{10}$$
$$6_{10} - \boxed{4_{10}} = 2_{10}$$
$$2_{10} - \boxed{2_{10}} = 0_{10}$$

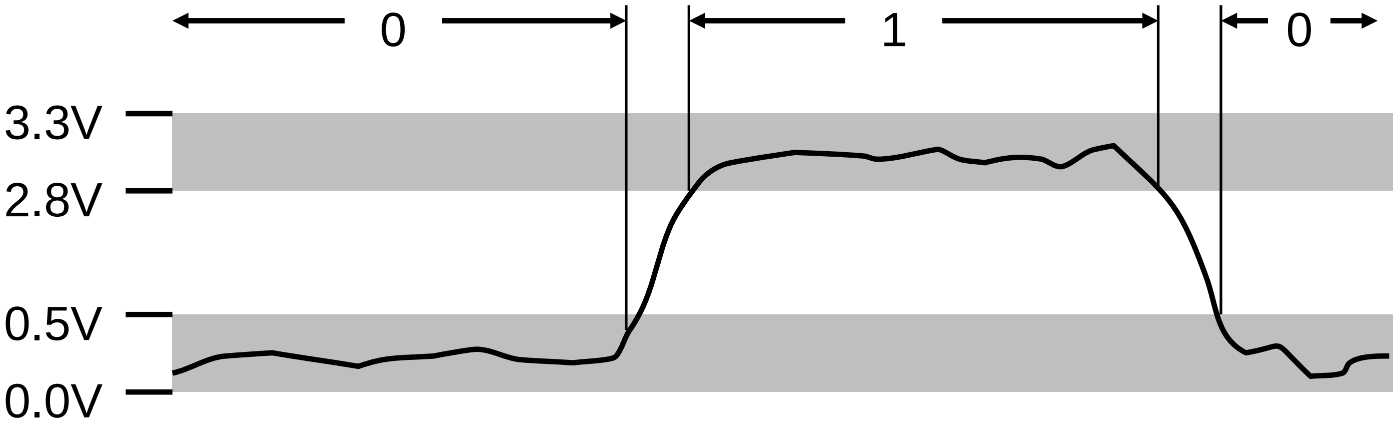
$$134_{10} = \underline{\mathbf{1}}\times 128 + 0\times 64 + 0\times 32 + 0\times 16 + 0\times 8 + \underline{\mathbf{1}}\times 4 + \underline{\mathbf{1}}\times 2 + 0\times 1$$

$$134_{10} = \underline{\mathbf{1}}\times 2^7 + 0\times 2^6 + 0\times 2^5 + 0\times 2^4 + 0\times 2^3 + \underline{\mathbf{1}}\times 2^2 + \underline{\mathbf{1}}\times 2^1 + 0\times 2^0$$

$$134_{10} = 10000110_2$$

Why computers use Base 2

- Simple electronic implementation
  - Easy to store with bi-stable elements
  - Reliably transmitted on noisy and inaccurate wires

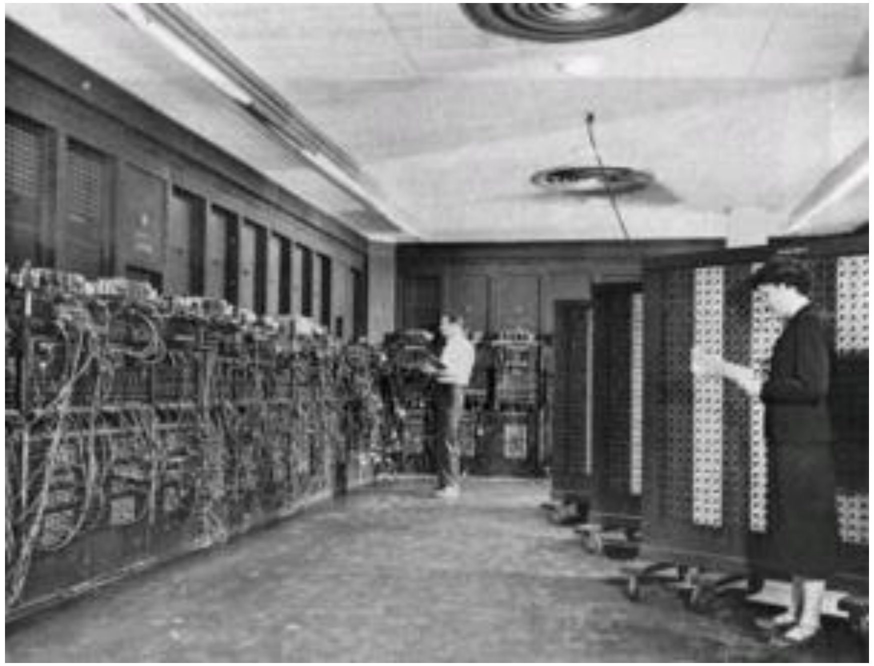


- Straightforward implementation of arithmetic functions

- (Pretty much) all computers use base 2

Why don't computers use Base 10?

- Because implementing it electronically is a pain
  - Hard to store
    - ENIAC (first general-purpose electronic computer) used 10 vacuum tubes / digit
  - Hard to transmit
    - Need high precision to encode 10 signal levels on single wire
  - Messy to implement digital logic functions
    - Addition, multiplication, etc.



Base 16: Hexadecimal

- Writing long sequences of 0s and 1s is tedious and error-prone
  - And takes up a lot of space on a page!
- So we'll often use base 16 (also called *hexadecimal*)

- Base 2 = 2 symbols (0, 1)
  Base 10 = 10 symbols (0-9)
  Base 16, need 16 symbols
  - Use letters A-F once we run out of decimal digits

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Base 16: Hexadecimal

- $16 = 2^4$, so every group of 4 bits becomes a hexadecimal digit (or *hexit*)
  - If we have a number of bits not divisible by 4, add 0s on the left (always ok, just like base 10)

0 0 1 0 | 1 0 0 1 | 0 1 1 1 | 1 0 1 1 ⟶ 0x297B

"0x" prefix = it's in hex

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Bytes

- A single bit doesn't hold much information
  - Only two possible values: 0 and 1
  - So we'll typically work with larger groups of bits
- For convenience, we'll refer to groups of 8 bits as ***bytes***
  - And usually work with multiples of 8 bits at a time
  - Conveniently, 8 bits = 2 hexits
- Some examples
  - 1 byte: 0b01100111 = 0x67
  - 2 bytes: 11000100 00101111_2 = 0xC42F

"0b" prefix = it's in binary

Practice problem

- **Convert 0x42 to decimal**

- Steps
    - Convert 0x42 to binary:

    - Convert binary to decimal:

Practice problem

- **Convert 0x42 to decimal**

- Steps
  - Convert 0x42 to binary:
    - 0x4 -> 0b0100 0x2 -> 0b0010 0x42 -> 0b 0100 0010
  - Convert binary to decimal:

Practice problem

- **Convert 0x42 to decimal**

- Steps
  - Convert 0x42 to binary:
    - 0x4 -> 0b0100   0x2 -> 0b0010   0x42 -> 0b 0100 0010
  - Convert binary to decimal:
    - $1*2^6 + 1*2^1 = 64 + 2 = 66$

Practice problem

- **Convert 0x42 to decimal**

- Critical thinking:
  - What are the maximum and minimum values?
    - Minimum 0 (0x00)
    - Maximum 255 (0xFF)

  - How big is 0x42 out of 0xFF?
    - ~25% (0x40, 0x80, 0xC0, 0x100)
    - So 255/4 ≈ 256/4 ≈ 64

Outline

- Bits and Bytes
- **Integer Encoding**
- C Type Bounds

These two lines of code are equivalent

```
char mychar = 97;

char mychar = ‘a’;
```

- Per the ASCII table, the character ‘a’ has a decimal value 97
  - The character value and decimal value are equivalent
  - These two are also equivalent

```
char diff = ‘c’ - ‘a’;

char diff = 99 - 97;
```

Big idea: bits can be used to represent anything

- Depending on the context, the bits **`11000011`** could mean
  - The number 195
  - The number -61
  - The number -1.1875
  - The value `True`
  - The character '├'
  - The **`ret`** x86 instruction

- You have to know the **context** to make sense of any bits you have!
  - People and software they write determine what the bits actually mean

Expressing C types in bits

- Two families of encodings to express those using bits
  - ***Unsigned*** encoding for unsigned integers
  - ***Two's complement*** encoding for signed integers

- Each encoding will use a fixed size (# of bits)
  - For a given machine
  - Size + encoding family determine which C type we're representing
  - Fixed size is because computers are finite!

Unsigned integer encoding

- Just write out the number in binary
  - Works for 0 and all positive integers

- Example: encode 104_{10} as an **unsigned** 8-bit integer
  - $104_{10} = 0 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$

$\Rightarrow$ **`01101000`**

$\Rightarrow$ **`0x68`**

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

(Binary To Unsigned)

Bounds of unsigned integers

- For a fixed width ***w***, a limited range of integers can be expressed
  - Smallest value (we will call ***UMin***):
    - all 0s bit pattern: 000…0, value of 0
  - Largest value (we will call ***UMax***):
    - all 1s bit pattern: 111...1, value of $2^w - 1$
    - $2^w - 1 = 1\times2^{w-1} + 1\times2^{w-2} + ... + 1\times2^1 + 1\times2^0 = 11111...$
- Maximum 8-bit number = 2^8-1 = 256-1 = 255

Two's complement encoding

- Good news: can represent both positive and negative numbers
- Bad news: need to make the encoding more complicated

- Plan:
  - Start with unsigned encoding, but make the largest power negative
  - Example: for 8 bits, most significant bit is worth -2^7 not $+2^7$

- To encode a negative integer
  - First, set the most significant bit to 1 to start with a big negative number
  - Then, add positive powers of 2 (the other bits) to "get back" to number we want

- Example: encode -6 as a 4-bit two's complement integer
  - $-6_{10} = 1 \times -2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \Rightarrow$ `0b1010` $\Rightarrow$ **`0xA`**

Two's complement examples

- Encode -100 as an 8-bit two's complement number

  - $-100_{10} =$ $1 \times -2^7 + 0 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$

    $-128 + 0 + 0 + 16 + 8 + 4 + 0 + 0$

    Problem becomes:
    encode +28 as a 7-bit unsigned number

  - -100_{10} = 0b10011100 = 0x9C

Two's Complement Shortcut

- **Shortcut:** determine positive version of number, flip it, and add one

  - 100_{10} = 0b01100100
  - Flipped = 0b10011011
  - Plus 1 = 0b10011100 = 0x9C

Sidebar: binary addition

```
 0b01          0b011
+0b01         +0b001
------        -------
 0b10          0b100
```

Interpreting binary signed values

- Converting binary to signed: $$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

  Sign bit

- Note: most significant bit still tells us sign!! 1 -> negative
  - Checking if a number is negative is just checking that top bit

- Note: there is only one zero value
  - 0b00000000 = 0 0b10000000 = -128

- -1: 0b111…1 = -1 (regardless of number of bits!)

Bounds of two's complement integers

- For a fixed width ***w***, a limited range of integers can be expressed
  - Smallest value, most negative (we will call ***TMin***):
    - 1 followed by all 0s bit pattern: 100…0 = -2^{w-1}
  - Largest value, most positive (we will call ***TMax***):
    - 0 followed by all 1s bit pattern: 01...1, value of $2^{w-1} - 1$
- Beware the asymmetry! Bigger negative number than positive

Unsigned & Signed Numeric Values

X	B2U(*X*)	B2T(*X*)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- **Equivalence**
  - Same encodings for non-negative values
- **Uniqueness**
  - Every bit pattern represents unique integer value
  - Each representable integer has unique bit encoding
- $\Rightarrow$ **Can Invert Mappings**
  - Can go from bits to number and back, and vice versa
  - $\mathrm{U2B}(x) = \mathrm{B2U}^{-1}(x)$
    - Bit pattern for unsigned integer
  - $\mathrm{T2B}(x) = \mathrm{B2T}^{-1}(x)$
    - Bit pattern for two’s complement integer

Practice + Break

- What range of integers can be represented with 5-bit two's complement?

  - A   -31 to +31
  - B   -15 to +15
  - C   0 to +31
  - D   -16 to +15
  - E   -32 to +31

Practice + Break

- What range of integers can be represented with 5-bit two's complement?

• A	-31 to +31	No asymmetry and 6-bits
• B	-15 to +15	No asymmetry
• C	0 to +31	Unsigned
• D	$\boxed{-16 \text{ to } +15}$	Correct
• E	-32 to +31	6-bits

Outline

- Bits and Bytes
- Integer Encoding
- **C Type Bounds**

Standard sizes of C types on modern (64-bit) computers

- 1 byte
  - char, unsigned char, signed char
  - bool
- 2 bytes
  - short, unsigned short, signed short
- 4 bytes
  - int, unsigned int, signed int
  - float
- 8 bytes
  - long, unsigned long, signed long
  - double
  - Every pointer type!

Ranges for different bit amounts

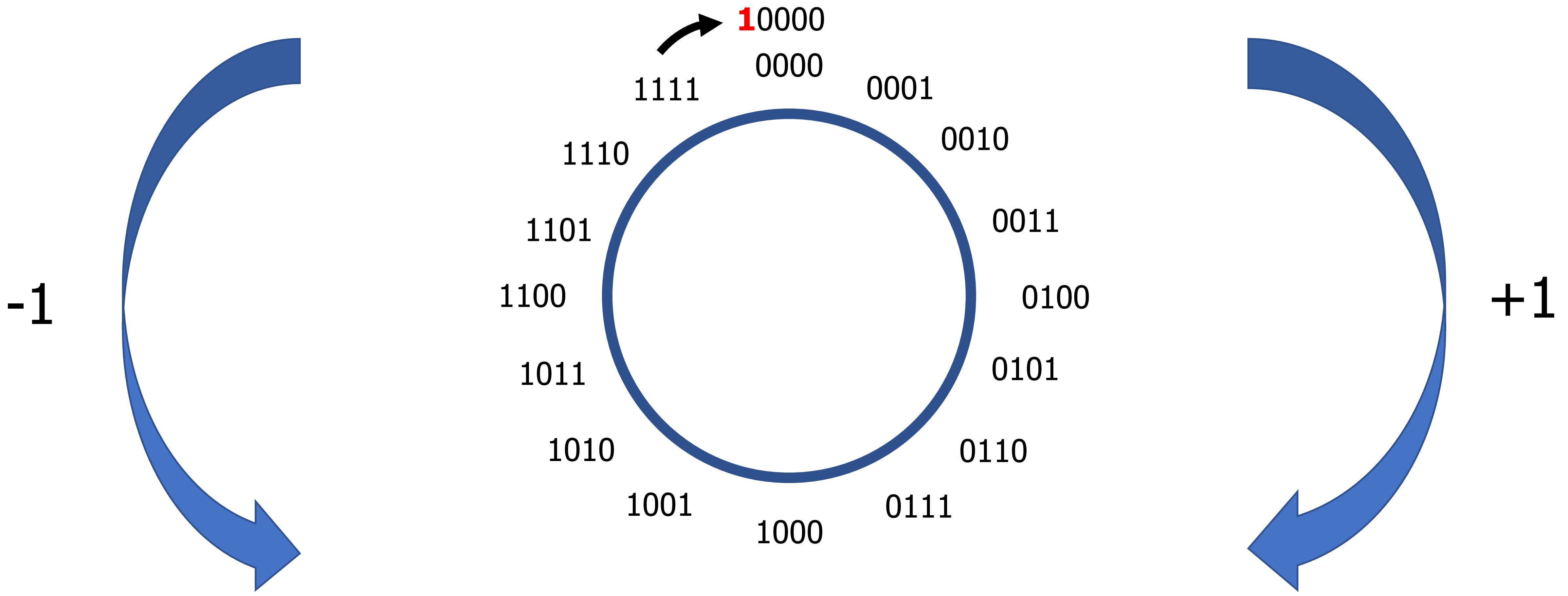
	W			
	8	16	32	64
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- Observations
  - $|TMin| = TMax + 1$
    - Asymmetric range
  - $UMax = 2 * TMax + 1$
- C Programming
  - #include <limits.h>
  - Declares constants, e.g.,
    - ULONG_MAX
    - LONG_MAX
    - LONG_MIN
  - Values are platform specific

Overflow

- What happens if you exceed the bound of a variable type?

Modulo behavior in binary numbers



Overflow

- What happens if you exceed the bound of a variable type?

- Unsigned Variables
  - They wrap!

```
char a = 255;
a++;
// a now equals 0

char b = 2;
b = b-5;
// b now equals 253
```

Overflow

- What happens if you exceed the bound of a variable type?

- Signed Variables
  - **UNDEFINED BEHAVIOR**

  - Often they wrap
  - But also the compiler can do anything it wants

Remember that overflow/underflow can occur in C

- Warning: programmers often fail to account for wrapping!
  - Sometimes it leads to unexpected behavior

Overflow example in the real world

- Dream Devourer
  - Special boss in the Nintendo DS edition
- Wanted to make it even more challenging
  - 32000 hit points
  - Takes *forever* to defeat
- Hit points stored as a 16-bit signed integer
  - Range: -32768 to +32767

Chrono Trigger signed overflow bug

- Solution: heal it
- Hit points go negative and it dies

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